TMA 4115 Matematikk 3

Lecture for FYMA

Alexander Schmeding

NTNU

23. March 2017

In today's lecture we will...

- Recap: Similarity of matrices
- use similarity to relate matrices to diagonal matrices
- complex eigenvalues and eigenvectors

Similarity of matrices

Similarity of matrices A and B

A is **similar** to B if there is an invertible matrix P with

$$PAP^{-1} = B \quad \text{(or equivalently } A = P^{-1}BP\text{)}$$

Recall that similar matrices have the same characteristic polynomial (hence the same eigenvalues).

- $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ are not similar.
- $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$ are similar, since

$$C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

We compute

- eigenvalues via the **characteristic polynomial** $\det(A - \lambda I)$.
- eigenvectors via $(A - \lambda I)\overrightarrow{x} = \overrightarrow{0}$ (for λ eigenvalue!)

Example: Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Calculate the roots of the **characteristic polynomial**

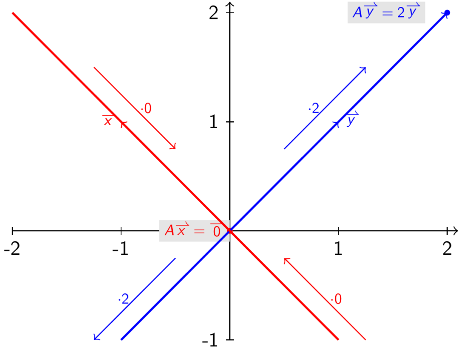
$$\det(A - \lambda I) = (1 - \lambda)^2 - 1 = \lambda^2 - 2\lambda = (0 - \lambda)(2 - \lambda).$$

Roots of the characteristic polynomial = eigenvalues.
$\lambda = 0$ and $\lambda = 2$.

Calculate eigenvectors for $\lambda = 0$ and $\lambda = 2$ via Gaussian elimination, i.e. find a non trivial solution of $(A - 0I)\overrightarrow{x} = \overrightarrow{0}$ and $(A - 2I)\overrightarrow{y} = \overrightarrow{0}$.

For example $\overrightarrow{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\overrightarrow{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Along the eigenvectors the matrix acts by scalar multiplication:



Observe that the eigenvectors even form a basis $\mathcal{B}$ for $\mathbb{R}^2$!

Observation: Matrix action in $\mathcal{B}$ coordinates

Recall $\mathcal{B}$-coordinates $[\overrightarrow{x}]_{\mathcal{B}}$, have invertible linear map ($\sim$ matrix) $P_{\mathcal{B}}$ with $P_{\mathcal{B}}[\overrightarrow{x}]_{\mathcal{B}} = \overrightarrow{x}$.Then

$$
\begin{tikzcd}
\overrightarrow{x} \arrow[r, "A\cdot"] \arrow[d, "P_{\mathcal{B}}^{-1}\cdot"] & A\overrightarrow{x} \\
{[\overrightarrow{x}]_{\mathcal{B}}} \arrow[r, "\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\cdot"] & {[A\overrightarrow{x}]_{\mathcal{B}}} \arrow[u, "P_{\mathcal{B}}\cdot"']
\end{tikzcd}
$$

$\to$ A acts on the $\mathcal{B}$-coordinates as a diagonal matrix!

Note $A = P_{\mathcal{B}} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} P_{\mathcal{B}}^{-1}$

Question:

Can we "transform" (all?) square matrices to diagonal matrices?

A a $n \times n$-matrix. Find a similar diagonal matrix (if possible).

1. Compute the eigenvalues of$A$,
2. Compute the eigenvectors associated to the eigenvalues. (Gaussian elimination!)
3. Check if we have n linear independent eigenvectors. If not: A is not diagonalisable!
   Hint: Eigenvectors of different eigenvalues are linearly independent.
4. Write eigenvectors as columns in a matrix P
5. Construct a diagonal matrix D whose diagonal entries are the eigenvalues corresponding to the columns in P
6. Then $A = PDP^{-1}$