TMA 4115 Matematikk 3

Lecture for FYMA

Alexander Schmeding

NTNU

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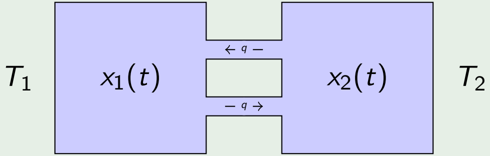
In today's lecture we discuss...

- Systems of linear differential equations

Kont 2012, Problem 7

Two water tanks, T_1 and T_2, each with volume $V = 100$ litre, are connected together with pipes as shown in the figure below.



The tanks are filled with salt water; $x_1(t)$ and $x_2(t)$ are the mass in grammes of salt in the respective tanks at time t. Salt water flows from tank T_1 to tank T_2, and equally from T_2 to T_1, at the rate $q = 1$ litres per second in each direction. We ignore the volume of the pipes, and assume instantaneous mixing of salt water [...] Let $x_1(0) = 100g$ and $x_2(0) = 0g$. At what time is $x_2(t) = 25g$?

Observation

Find two unknown functions. Varying time from t to $t+\Delta t$ we see

$$x_1(t+\Delta t) = x_1(t) - \frac{\Delta t}{100}x_1(t) + \frac{\Delta t}{100}x_2(t)$$
$$x_2(t+\Delta t) = x_2(t) + \frac{\Delta t}{100}x_1(t) - \frac{\Delta t}{100}x_2(t)$$

For $\Delta t \to 0$ we see

$$x_1'(t) = -\frac{1}{100}x_1(t) + \frac{1}{100}x_2(t)$$
$$x_2'(t) = \frac{1}{100}x_1(t) - \frac{1}{100}x_2(t)$$

This combines linear systems and linear differential equations!

A **system of linear differential equations** is given by

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_2 + \cdots + a_{2n}x_n \\ \vdots & \qquad\quad \vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

$x_1, \ldots, x_n$ functions with derivatives $x_1', \ldots, x_n'$, a_{ij} constant.

A **solution** to the system are differentiable functions $x_1, \ldots, x_n$ which satisfy the equations.

With $A = \left[a_{ij}\right]_{1 \leq i,j \leq n}$ rewrite the system as:

$$\overrightarrow{x}'(t) = A\overrightarrow{x}(t)$$

Strategy to solve systems of linear differential equations

Solve the system $\overrightarrow{x}'(t) = A\overrightarrow{x}$ **if** A is diagonalizable:

1. Compute eigenvalues $\{ \lambda_1, \ldots, \lambda_k \}$ with eigenvectors $\{ \overrightarrow{v}_1, \ldots, \overrightarrow{v}_k \}$ of A.
2. General solution
$$\overrightarrow{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \overrightarrow{v}_i \tag{1}$$
3. Complex eigenvalue $\lambda = a + ib$: Solution is complex and we have to construct a real valued solution. Omit the part corresponding to $\overline{\lambda}$ and replace the λ part in (1) by
$$(r(\cos(bt)\overrightarrow{v}_r - \sin(bt)\overrightarrow{v}_i) + s(\cos(bt)\overrightarrow{v}_i + \sin(bt)\overrightarrow{v}_r))e^{at},$$
for $r, s \in \mathbb{R}$
4. Fix initial conditions, i.e. choose c_i in (1) with $\overrightarrow{x}(0) = \sum_{i=1}^{n} c_i \overrightarrow{v}_i$.