TMA 4115 Matematikk 3

Lecture 3 for FYMA

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Topics in todays lecture...

- Recap: Operations on complex numbers
- Recap: Roots of complex numbers
- Functions using complex numbers

Overview: Operations on complex numbers

For $w = a + ib = |w|(\cos(\theta) + i\sin(\theta))$,
$z = x + iy = |z|(\cos(\psi) + i\sin(\psi))$ we can compute...

Sum $w + z$

$w + z = (x + y) + i(b + y))$

Product $w \cdot z$ and powers z^n for $n \in \mathbb{N}$

$|w||z|(\cos(\theta + \psi) + i\sin(\theta + \psi))$ and $z^n = |z|^n(\cos(n\psi) + i\sin(n\psi))$

Conjugate $\overline{w}$

$\overline{w} = a - ib$

Reciprocal z^{-1} and quotient $\frac{w}{z}$ for $z \neq 0$

$z^{-1} = \frac{\overline{z}}{|z|^2}$ and $\frac{w}{z} = \frac{w \cdot \overline{z}}{|z|^2}$

Roots of complex numbers

nth roots of a complex number

For $z \in \mathbb{C}$ find the $w \in \mathbb{C}$ (*nth roots*) which solve

$$w^n = z. \tag{1}$$

By the formula for multiplication and (1) we have

$$|w| = \sqrt[n]{|z|} \text{ and } \arg(w) = \frac{\arg(z)}{n}$$

Let $z = r(\cos(\theta) + i\sin(\theta))$, $\theta = \operatorname{Arg}(z)$ and $n \in \mathbb{N}$.

We call

$$z_0 = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right)$$

the *principal nth root of z*.

Roots for $z = r(\cos(\theta) + i\sin(\theta)) \neq 0$ and $n \in \mathbb{N}$

There are n distinct nth roots which can be computed via

$$\begin{aligned}
z_0 &= \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right) \\
z_1 &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+2\pi}{n}\right) + i\sin\left(\frac{\theta+2\pi}{n}\right)\right) \\
&\vdots \\
z_{n-1} &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+2(n-1)\pi}{n}\right) + i\sin\left(\frac{\theta+2(n-1)\pi}{n}\right)\right)
\end{aligned}$$

General formula: For $k = 0, 1, \ldots, n-1$

$$z_k = \sqrt[n]{r}\left(\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right)$$

Examples for nth roots of z

$z = 1,\ n = 4$ Polar form $z = 1(\cos(0) + i\sin(0))$

$$\begin{aligned} z_0 &= 1(\cos(0) + i\sin(0)) = 1 \quad \text{(principal root)} \\ z_1 &= 1(\cos(\tfrac{0+2\pi}{4}) + i\sin(\tfrac{0+2\pi}{4})) = i \\ z_2 &= 1(\cos(\tfrac{0+2\pi\cdot 2}{4}) + i\sin(\tfrac{0+2\pi\cdot 2}{4})) = -1 \\ z_3 &= 1(\cos(\tfrac{0+2\pi\cdot 3}{4}) + i\sin(\tfrac{0+2\pi\cdot 3}{4})) = -i \end{aligned}$$

$z = i,\ n = 3$ Polar form $z = 1(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}))$

$$\begin{aligned} z_0 &= 1(\cos(\tfrac{\pi}{6}) + i\sin(\tfrac{\pi}{6})) = \tfrac{\sqrt{3}}{2} + i\tfrac{1}{2} \quad \text{(principal root)} \\ z_1 &= 1(\cos(\tfrac{\pi}{6} + \tfrac{2\pi}{3}) + i\sin(\tfrac{\pi}{6} + \tfrac{2\pi}{3})) = -\tfrac{\sqrt{3}}{2} + i\tfrac{1}{2} \\ z_2 &= 1(\cos(\tfrac{\pi}{6} + \tfrac{4\pi}{3}) + i\sin(\tfrac{\pi}{6} + \tfrac{4\pi}{3})) = -i \end{aligned}$$

Real functions

Recall that a *real function* is a triple

$$f\colon U \to V, \quad x \mapsto f(x)$$

where

$$\begin{aligned} U \subseteq \mathbb{R} &\text{ the } \textit{domain} \text{ (i.e. the numbers we apply } f \text{ to)} \\ V \subseteq \mathbb{R} &\text{ the } \textit{codomain} \text{ (must contain all values of } f\text{)} \\ x \mapsto f(x) &\text{ a } \textit{rule} \text{ assigning to each } x \text{ an element } f(x) \end{aligned}$$

Examples:

$$g\colon (0,1) \to (0,\infty), \quad x \mapsto \frac{1}{x}$$
$$\sin\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sin(x)$$

Here (a, b) is the open interval from a to b.

Complex functions functions

A *complex function* is a triple

$$f\colon U \to V, \quad x \mapsto f(x)$$

where

$$\begin{aligned} U &\subseteq \mathbb{C} \text{ the } \textit{domain} \text{ (i.e. the numbers we apply } f \text{ to)} \\ V &\subseteq \mathbb{C} \text{ the } \textit{codomain} \text{ (must contain all values of } f) \\ x \mapsto f(x) &\text{ a } \textit{rule} \text{ assigning to each } x \text{ an element } f(x) \end{aligned}$$

Every complex function f can be written as $f = \mathrm{Re}(f) + i\mathrm{Im}(f)$ where $\mathrm{Re}(f)(z) := \mathrm{Re}(f(z))$ and $\mathrm{Im}(f)(z) := \mathrm{Im}(f(z))$.

Example:

$$g\colon \mathbb{C} \to \mathbb{C}, \quad a + ib \mapsto 42(a+ib)^2 = 42(a^2 - b^2 + 2iab)$$

$\mathrm{Re}(g)(a+ib) = 42(a^2 - b^2)$, $\mathrm{Im}(g)(a+ib) = 84ab$.