TMA 4115 Matematikk 3

Lecture 4 for FYMA

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Topics in todays lecture...

- Recap: The complex exponential function
- 2nd order differential equations

Complex functions

A *complex function* $f\colon U \to V, \quad x \mapsto f(x)$ consists of $U \subseteq \mathbb{C}$ (domain), $V \subseteq \mathbb{C}$ (codomain) and a rule $x \mapsto f(x)$.

Complex polynomials

$p\colon \mathbb{C} \to \mathbb{C}, \quad p(z) = \sum_{k=0}^{n} a_k z^k$ (complex) *polynomial*, where $a_0, \ldots, a_n \in \mathbb{C}$ and $n \in \mathbb{N}$.

$w \in \mathbb{C}$ is a root of p if $p(w) = 0$. For $p(z) = az^2 + bz + c$ we can find roots by the quadratic formula.

The complex exponential function

$\exp\colon \mathbb{C} \to \mathbb{C},\ \exp(x+iy) = e^x(\cos(y) + i\sin(y))$ write $e^z := \exp(z)$

Properties of exp for $z = x + iy$

(a) $\exp(a + 0i) = e^a$, (exp for real numbers),

(b) $|e^z| = e^x, \quad \arg(e^z) = y$,

(c) $\mathrm{Re}(e^z) = e^x\cos(y)$ and $\mathrm{Im}(e^z) = e^x\sin(y)$,

(d) $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for $z_1, z_2 \in \mathbb{C}$,

(e) $\frac{1}{e^z} = e^{-z} \quad \overline{\exp(z)} = \exp(\overline{z})$,

Can use exp to write polar coordinates:

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

We know how to solve the differential equations

$$y'(t) = \frac{d}{dt}y(t) = f(y,t)$$

if the function f is "nice".

This is a *first order differential equation*, because it **only involves the first derivative** of y.

First order differential equations are "simple". The (physical) world is complicated, whence it can not only be described by first order equations. We need more complicated equations!

3.1 Newtons second law

Newtons second law

The *acceleration* a of a body with *mass* m is proportional to the *net force* F via

$$F = ma \tag{1}$$

Question: What is the displacement $y(t)$ of the body from a reference point?

Acceleration a is rate of change of velocity v, i.e.

$$a = \frac{d}{dt}v = v' \tag{2}$$

Velocity v is rate of change of the displacement, i.e.

$$v = \frac{d}{dt}y = y' \tag{3}$$

Thus (2) becomes $a = y''$.

What is the displacement $y(t)$ of the body from a reference point?

Finally, F depends on time, displacement and velocity, thus

$$F(t, y, v) = F(t, y, y') \quad \text{(using (3))}$$

With Newtons second law (1) obtain

$$F(t, y, y') = my''.$$

This is a **second order differential equation** since it involves derivatives of y of up to second order.

3.2 Definition

A *second order differential equation* is an equation of the form

$$\frac{d^2}{dt^2}y(t) = f\left(y, \frac{d}{dt}y, t\right) \tag{4}$$

where f is a given function.

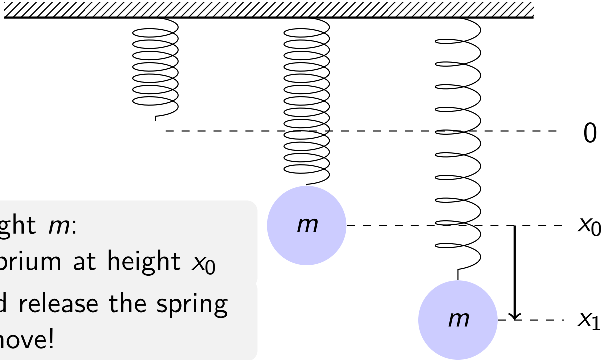
A *solution* to (4) is a function y which is twice continuously differentiable and satisfies (4).

Usually we write $y' = \frac{d}{dt}y(t)$ and $y'' = \frac{d^2}{dt^2}y(t)$, thus (4) reads:

$$y'' = f(t, y, y') \tag{5}$$

3.3 The vibrating spring

We consider a spring suspended from a beam:



Attach weight m:
New equilibrium at height x_0

Stretch and release the spring
. . . it will move!

Forces acting on weight in motion:

damping force $D(v)$ (depends on velocity $v = x'$),
external force $F(t)$, **Restoring force** $R(x)$ and **gravity** mg.

A model for the position $x(t)$ of the spring

Newtons second law (1) for the spring

$$\begin{aligned} ma &= \text{total force acting on the weight} \\ &= R(x) + mg + D(v) + F(t) \end{aligned} \tag{6}$$

Velocity $v = x'$ and acceleration $a = v' = x''$, thus (6) becomes

$$mx'' = R(x) + mg + D(x') + F(t). \tag{7}$$

Hooke's law (valid for some springs)

$R(x) = -kx$ for $k > 0$ constant and small x.

Assuming Hooke's law, (7) becomes

$$x'' = -\frac{k}{m}x + g + \frac{D(x') + F(t)}{m}. \tag{8}$$

Is there a solution for every 2nd order differential equation?

3.4 Theorem

Let p, q and g be continuous functions on the interval (α, β). Fix $t_0 \in (\alpha, \beta)$ and $y_0, y_1 \in \mathbb{R}$. There is a unique function $y\colon (\alpha, \beta) \to \mathbb{R}$ which solves

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), & \text{for } t \in (\alpha, \beta) \\ y(t_0) = y_0, y'(t_0) = y_1 \end{cases} \tag{9}$$

3.5 Remark

The solution of (9) is defined on all of (α, β)
We need $y(t_0) = y_0, y'(t_0) = y_1$ to get a unique solution.

Open Problem: How to find a solution?