TMA 4115 Matematikk 3

Lecture 5 for FYMA

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Topics in todays lecture...

- Recap: (linear) 2nd order differential equations
- Linear independence of functions (New test: The Wronskian)
- Linear differential equations with constant coefficients

Want to solve second order differential equations

$$y'' = f(t, y, y').$$

Linear (second order) differential equations are of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q and g depend only on t (not on y).

If the forcing term g is 0, the equation is **homogeneous**, otherwise **inhomogeneous**.

An **initial value problem** (IVP) is a differential equation with enough initial values to specify a solution.

$u, v\colon (\alpha,\beta) \to \mathbb{R}$ are **linearly independent** (on (α,β)) if there is no $C \in \mathbb{C}$ with $u(t) = Cv(t)$ for all $t \in (\alpha,\beta)$.

Theorem (A proof is in your book p. lxii)

Let y_1, y_2 be linearly independent solutions to

$$y'' + p(t)y' + q(t)y = 0$$

The **general solution**[a] to the differential equation is

$$y(t) = Ay_1(t) + By_2(t) \text{ where } A, B \in \mathbb{C}.$$

[a] this means all solutions are of the form $Ay_1 + By_2$

We call two linearly independent solutions for a second order homogeneous linear equation a **fundamental set** of solutions.

Strategy to solve homogeneous linear differential equations

$$y'' + p(t)y' + q(t)y = 0$$

Obtain general solution

- Find two solutions u, v
- Check that u, v are linearly independent
- General solution $Au + Bv$

Solve IVP $y'' + p(t)y' + q(t)y = 0, y(t_0) = y_0, y'(t_0) = y_1$

- Need general solution to $y'' + p(t)y' + q(t)y = 0$
- Use $y(t_0) = y_0$ and $y'(t_0) = y_1$ to determine A and B

Tests for linear independence

- **Inspection**

$u(t) = \sin(t)$ and $v(t) = \cos(t)$

For $t = 0$ have $\sin(0) = 0$ and $\cos(0) = 1$.Thus we can only have $C = 0$ and $\sin(t) = 0\cos(t)$. Does not work since $\sin \neq 0$.Thus sin and cos are linearly independent.

Now: A new test for linear independence: The Wronskian.

Problem: How to find any solution?

We know what to do if we already found solutions to a linear homogeneous equation. However, how do we find these solutions?

Goal: Construct solutions for simpler homogeneous linear equations, i.e. equations with <u>constant</u> coefficients.

$$(\text{i.e. } p \equiv \text{const}, q \equiv \text{const})$$

Idea: Consider

$$y' + qy = 0, \quad q \in \mathbb{C}$$

We know that $y(t) = Ce^{-qt}$ solves the equation for all $C \in \mathbb{C}$.
Try $y(t)$ as a solution to a second order homogeneous equation.

Equation $y'' + py' + qy = 0$ Polynomial: $\lambda^2 + p\lambda + q = 0$

Case 1 $p^2 - 4q > 0$, i.e. two distinct, real roots λ_1 and λ_2. Fundamental set of solutions:

$$y_1(t) = e^{\lambda_1 t} \quad y_2(t) = e^{\lambda_2 t}$$

Case 2 $p^2 - 4q < 0$, we have two distinct, complex roots $\lambda_1 = a + ib$ and $\overline{\lambda_1}$. Fundamental set of (real valued) solutions:

$$y_1(t) = e^{at}\cos(bt) \quad y_2(t) = e^{at}\sin(bt)$$

Case 3 $p^2 - 4q = 0$, there is one repeated real root λ_1. Fundamental set of solutions:

$$y_1(t) = e^{\lambda_1 t}, \quad y_2(t) = te^{\lambda_1 t}$$