TMA 4115 Matematikk 3

Lecture 6 for FYMA

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Topics in todays lecture...

- Recap: Differential equations with constant coefficients
- Inhomogeneous differential equations
- The Method of undetermined coefficients

If $p, q \in \mathbb{R}$ constant and g a function, we call

$$y'' + py' + qy = g(t) \tag{$\star$}$$

(linear inhomogeneous) differential equation with constant coefficients.

Associated homogeneous equation to $(\star)$

$$y'' + py' + qy = 0 \tag{$\star\star$}$$

Last lecture: How to solve homogeneous equations

- Compute *characteristic polynomial* $\lambda^2 + p\lambda + q$ for $(\star)$
- Find the roots of the polynomial (*characteristic roots*)
- Have three cases depending on roots

Solutions for equations with constant coefficients

Case 1 $p^2 - 4q > 0$, i.e. two distinct, real roots λ_1 and λ_2. Fundamental set of solutions for $(\star)$:

$$y_1(t) = e^{\lambda_1 t} \quad y_2(t) = e^{\lambda_2 t}$$

Case 2 $p^2 - 4q < 0$, i.e. two distinct, complex roots $\lambda_1 = a + ib$ and $\overline{\lambda_1}$. Fundamental set of (real valued) solutions for $(\star)$:

$$y_1(t) = e^{at} \cos(bt) \quad y_2(t) = e^{at} \sin(bt)$$

Case 3 $p^2 - 4q = 0$, i.e. one repeated real root λ_1. Fundamental set of solutions for $(\star)$:

$$y_1(t) = e^{\lambda_1 t}, \quad y_2(t) = te^{\lambda_1 t}$$

Harmonic motion

$$y'' + 2cy' + \omega_0^2 y = f(t) \qquad \textbf{harmonic motion} \qquad (1)$$

with $c, \omega_0 \in \mathbb{R}$. We call

c *dampening parameter*

ω_0 *natural frequency*

Roots of $\lambda^2 + 2c\lambda + \omega_0^2$ are $\lambda_{1,2} = -c \pm \sqrt{c^2 - \omega_0^2}$.

Example: The spring equation (Chapter 3)

$$my'' = -ky - \mu y' + F(t) \qquad k, \mu > 0$$

or equivalently

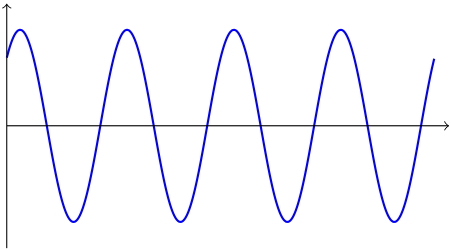
$$y'' + \frac{\mu}{m} y' + \frac{k}{m} y = \frac{F(t)}{m} \qquad (2)$$

Consider now $\mu = 0 = F(t)$ (simple case)

Simple Harmonic motion (i.e. $\mu = 0 = F(t)$)

Have $\lambda^2 + \frac{k}{m} = 0$ with complex roots $\pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$. General (real) solution of (2) is

$$\begin{aligned} y(t) &= Ae^{0t}\cos(\omega_0 t) + Be^{0t}\sin(\omega_0 t) \\ &= A\cos(\omega_0 t) + B\sin(\omega_0 t) \qquad A, B \in \mathbb{R} \end{aligned}$$



plotted for $A, B = 1$, $\omega_0 = 4$

No Damping or forcing: Solution oscillates with natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$

Inhomogeneous equations

A solution y_p to $y'' + p(t)y' + q(t)y = g(t)$ is called *particular solution*.

5.1 Theorem

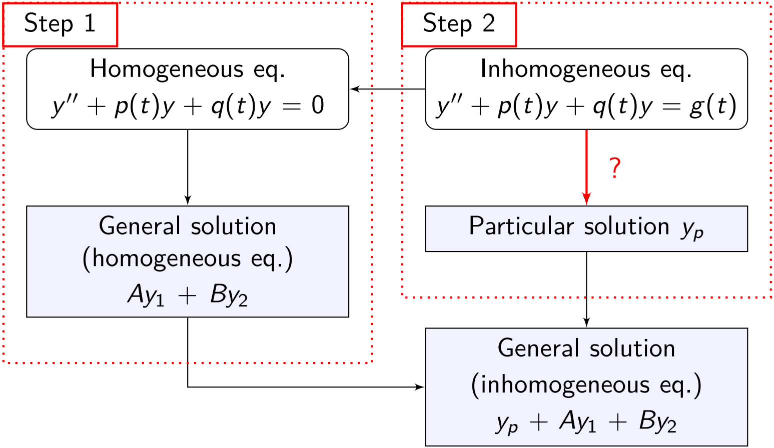
Let $y_p(t)$ be a particular solution and y_1, y_2 a fundamental system for the associated homogeneous equation

$$y'' + py' + qy = 0.$$

Then the general solution to the inhomogeneous equation is

$$y(t) = y_p(t) + Ay_1(t) + By_2(t) \quad A, B \in \mathbb{R}(\text{or } \mathbb{C}).$$

Solution to $y'' + p(t)y' + q(t)y = g(t)$



Problem: How to find a particular solution?

We now introduce the **method of undetermined coefficients**.

Idea: Guess a particular solution based on the forcing term.

Guideline for the method

If the form of the forcing term f replicates under differentiation, look for a solution with the same form.

Overwiew: The method of undetermined coefficients

Forcing term $f(t)$	Trial solution	Comment
e^{rt}	ae^{rt}	
$\cos(\omega t)$ or $\sin(\omega t)$	$a\cos(\omega t) + b\sin(\omega t)$	
$P(t)$ Polynomial	$p(t)$ Polynomial	$P(t)$ and $p(t)$ have same degree
Example: t^2	$at^2 + bt + c$	

Problem: If the trial solution is a solution of the homogeneous equation the above method does not work!

Solution: Try multiplying the trial solution with t, if that does not work multiply by t again.