TMA 4115 Matematikk 3

Lecture 7 for FYMA

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Topics in todays lecture...

- Recap: Method of undetermined coefficients
- Trigonometric forcing terms: “The complex method”
- Example: Forced harmonic motion
- Variation of Parameters

A general solution to the inhomogeneous equation

$$y'' + p(t)y' + q(t)y = f(t)$$

is of the form

$$y(t) = y_p(t) + Ay_1(t) + By_2(t), A, B \in \mathbb{C}.$$

Here

1. y_1, y_2 are a fundamental system for the associated homogeneous system
2. y_p is a particular solution to the inhomogeneous system

If the forcing term $f(t)$ is “nice” we can use the *method of undetermined coefficients* to compute y_p in step 2.

Method of undetermined coefficients

Guideline for the method

If the form of the forcing term f replicates under differentiation, look for a solution with the same form (with a parameter).

Complex method for trigonometric terms

Solve

$$y'' + p(t)y' + q(t)y = f(t)$$

for forcing term $f(t) = A\cos(wt)$ or $f(t) = A\sin(wt)$ with $A \in \mathbb{R}$.

Idea: $A\cos(wt) = \mathrm{Re}(Ae^{wit})$ and $A\sin(wt) = \mathrm{Im}(Ae^{wit})$.

Strategy

- Find a particular solution z_p of $z'' + p(t)z' + q(t)z = Ae^{wit}$
- The real part of z_p solves the equation for $y'' + p(t)y' + q(t)y = Acos(t)$ (imaginary part yields solution for forcing term $A\sin(wt)$)

5.8 Example $y'' + 2y' - 3y = 5\sin(3t)$

Solving $z'' + 2z' - 3z = 5e^{3it}$ we get $y_p(t) = -\frac{2+i}{6}e^{3it}$.

$$\begin{aligned} -\frac{2+i}{6}e^{3it} &= \frac{1}{6}(-2\cos(3t) - i\cos(3t) - 2i\sin(3t) - i^2\sin(3t)) \\ &= -\frac{1}{3}\cos(3t) + \frac{1}{6}\sin(3t) - i(\frac{1}{6}\cos(3t) + \frac{1}{3}\sin(3t)) \end{aligned}$$

$y_p(t) = \mathrm{Im}(-\frac{2+i}{6}e^{3it}) = -\frac{1}{6}\cos(3t) - \frac{1}{3}\sin(3t))$
(same as in Example 5.3)

Forced harmonic motion: Undampened case

Consider a forced harmonic motion

$$y'' + 2cy' + \omega_0^2 y = A\cos(\omega t), \qquad A \neq 0 \qquad (\star)$$

with $c, \omega_0, \omega \in \mathbb{R}$ and

c, the dampening parameter

ω_0, the natural frequency

ω, the driving frequency

Assume that $c = 0$, i.e. no damping in the harmonic motion.

Fundamental set of solutions for $(\star)$ and $c = 0$ is

$$y_1(t) = \cos(\omega_0 t), \qquad y_2(t) = \sin(\omega_0 t)$$

Case 1: Solve $(\star)$ for $\omega \neq \omega_0, c = 0$

Trial solution:

$y_p(t) = a\cos(\omega t) + b\sin(\omega t)$.

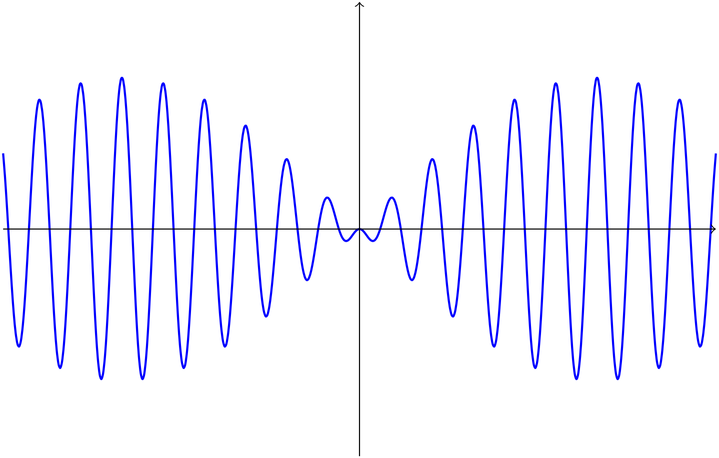
Insert in $(\star)$ and solve to find $a = \frac{A}{\omega^2 - \omega_0^2}$, $b = 0$ whence

$$y(t) = \frac{A}{\omega^2 - \omega_0^2}\cos(\omega t) + r\cos(\omega_0 t) + s\sin(\omega_0 t), \quad r, s \in \mathbb{R}$$

is the general solution to $(\star)$.
Solving the IVP $(\star)$, $y(0) = 0, y'(0) = 0$ yields

$$y(t) = \frac{A}{\omega^2 - \omega_0^2}(\cos(\omega t) - \cos(\omega_0 t))$$

Example plot: $A = 23, \omega = 12$ and $\omega_0 = 11$



Case $\omega_0 \neq \omega$: Fast oscillation with slowly varying amplitude

Case 2: $\omega = \omega_0 . c = 0$. Problem: trial solution solves homogeneous equation.

Trial solution

$t(a\cos(\omega t) + b\sin(\omega t))$ (Multiply trial solution from Case 1 by t)

Solve to find

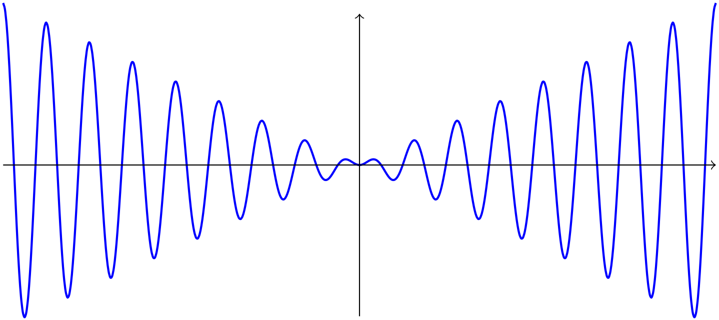
$$y(t) = \frac{A}{2\omega_0} t \sin(\omega_0 t) + r\cos(\omega_0 t) + s\sin(\omega_0 t), \;\; r, s \in \mathbb{R}$$

the general solution to $(\star)$. With initial conditions $y(0) = 0, y'(0) = 0$ this reduces to

$$y(t) = \frac{A}{2\omega_0} t \sin(\omega_0 t)$$

Example plot: $A = 1, \omega = \omega_0 = 11$

The solution $y(t) = \frac{t}{22}\sin(11t)$



Case $\omega_0 = \omega$: The amplitude grows linearly with time

6. Inhomogeneous equations: Variation of parameters

We want to construct solutions via "variation of parameters" for

$$y'' + p(t)y' + q(t)y = f(t) \tag{1}$$

Note: We do not assume constant coefficients here!

Idea: To find a particular solution for $y' + q(t)y = f(t)$ we set

$$y_p(t) = v(t)y_h(t)$$

where y_h is a solution for the associated homogeneous equation and v is an unknown function. Differentiate to determine v. Copy this idea using a fundamental set of solutions y_1, y_2.

Summary of the method

- We need a fundamental set y_1, y_2 of solutions to $y'' + p(t)y' + q(t)y = 0$.
- Define $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ with unknown functions v_1, v_2
- Solve

$$v_1(t) = \int \frac{-y_2(t)f(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}dt$$
$$v_2(t) = \int \frac{y_1(t)f(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}dt$$

to obtain y_p.

Alternatively, if you can't remember the formula: Derive it as explained in the lecture **6.1**.

We want to find a general solution to the inhomogeneous equation

$$y'' + py' + qy = f(t) \qquad (\diamond)$$

and solve the IVP $\begin{cases} y'' + py' + qy = f(t) \\ y(t_0) = y_0, y'(t_0) = y_1. \end{cases}$

1. Construct a fundamental set of solutions y_1, y_2 for $y'' + py' + qy = 0$,
2. Find a particular solution $y_p(t)$ for $(\diamond)$,
3. **General solution: $y(t) = y_p(t) + Ay_1(t) + By_2(t), A, B \in \mathbb{R}$**
4. Insert numbers in general solution to determine A and B.

If $f(t)$ is "nice" we can use the *method of undetermined coefficients* to compute y_p in step 2.