Kapittel 11.4

PROBLEM SET 11.3

1–9 EVEN AND ODD FUNCTIONS

Are the following functions even, odd, or neither even nor odd?

1. $|x|$, $x^2 \sin nx$, $x + x^2$, $e^{-|x|}$, $\ln x$, $x \cosh x$

2. $\sin(x^2)$, $\sin^2 x$, $x \sinh x$, $|x^3|$, $e^{\pi x}$, xe^x, $\tan 2x$, $x/(1 + x^2)$

Are the following functions, which are assumed to be periodic of period 2π, even, odd, or neither even nor odd?

3. $f(x) = x^3 \quad (-\pi < x < \pi)$

4. $f(x) = x^2 \quad (-\pi/2 < x < 3\pi/2)$

5. $f(x) = e^{-4x} \quad (-\pi < x < \pi)$

6. $f(x) = x^3 \sin x \quad (-\pi < x < \pi)$

7. $f(x) = x|x| - x^3 \quad (-\pi < x < \pi)$

8. $f(x) = 1 - x + x^3 - x^5 \quad (-\pi < x < \pi)$

9. $f(x) = 1/(1 + x^2)$ if $-\pi < x < 0$, $f(x) = -1/(1 + x^2)$ if $0 < x < \pi$

10. PROJECT. Even and Odd Functions. (a) Are the following expressions even or odd? Sums and products of even functions and of odd functions. Products of even times odd functions. Absolute values of odd functions. $f(x) + f(-x)$ and $f(x) - f(-x)$ for arbitrary $f(x)$.

(b) Write e^{kx}, $1/(1 - x)$, $\sin(x + k)$, $\cosh(x + k)$ as sums of an even and an odd function.

(c) Find all functions that are both even and odd.

(d) Is $\cos^3 x$ even or odd? $\sin^3 x$? Find the Fourier series of these functions. Do you recognize familiar identities?

11–16 FOURIER SERIES OF EVEN AND ODD FUNCTIONS

Is the given function even or odd? Find its Fourier series. Sketch or graph the function and some partial sums. (Show the details of your work.)

11. $f(x) = \pi - |x| \quad (-\pi < x < \pi)$

12. $f(x) = 2x|x| \quad (-1 < x < 1)$

13. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

14. $f(x) = \begin{cases} \pi e^{-x} & \text{if } -\pi < x < 0 \\ \pi e^{x} & \text{if } 0 < x < \pi \end{cases}$

15. $f(x) = \begin{cases} 2 & \text{if } -2 < x < 0 \\ 0 & \text{if } 0 < x < 2 \end{cases}$

16. $f(x) = \begin{cases} 1 - \frac{1}{2}|x| & \text{if } -2 < x < 2 \\ 0 & \text{if } 2 < x < 6 \end{cases} \quad (p = 8)$

17–25 HALF-RANGE EXPANSIONS

Find (a) the Fourier cosine series, (b) the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. (Show the details of your work.)

17. $f(x) = 1 \quad (0 < x < 2)$

18. $f(x) = x \quad (0 < x < \frac{1}{2})$

19. $f(x) = 2 - x \quad (0 < x < 2)$

20. $f(x) = \begin{cases} 0 & (0 < x < 2) \\ 1 & (2 < x < 4) \end{cases}$

21. $f(x) = \begin{cases} 1 & (0 < x < 1) \\ 2 & (1 < x < 2) \end{cases}$

22. $f(x) = \begin{cases} x & (0 < x < \pi/2) \\ \pi/2 & (\pi/2 < x < \pi) \end{cases}$

23. $f(x) = x \quad (0 < x < L)$

24. $f(x) = x^2 \quad (0 < x < L)$

25. $f(x) = \pi - x \quad (0 < x < \pi)$

26. Illustrate the formulas in the proof of Theorem 1 with examples. Prove the formulas.

11.4 Complex Fourier Series. *Optional*

In this optional section we show that the Fourier series

$$(1) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

can be written in complex form, which sometimes simplifies calculations (see Example 1, on page 498). This complex form can be obtained because in complex, the exponential function e^{it} and $\cos t$ and $\sin t$ are related by the basic **Euler formula** (see (11) in Sec. 2.2)

$$(2) \qquad e^{it} = \cos t + i \sin t. \qquad \text{Thus} \qquad e^{-it} = \cos t - i \sin t.$$

Conversely, by adding and subtracting these two formulas, we obtain

$$\text{(3)} \qquad \text{(a)} \quad \cos t = \frac{1}{2}(e^{it} + e^{-it}), \qquad \text{(b)} \quad \sin t = \frac{1}{2i}(e^{it} - e^{-it}).$$

From (3), using $1/i = -i$ in $\sin t$ and setting $t = nx$ in both formulas, we get

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= \frac{1}{2}\, a_n(e^{inx} + e^{-inx}) + \frac{1}{2i}\, b_n(e^{inx} - e^{-inx}) \\ &= \frac{1}{2}\,(a_n - ib_n)e^{inx} + \frac{1}{2}\,(a_n + ib_n)e^{-inx}. \end{aligned}$$

We insert this into (1). Writing $a_0 = c_0$, $\frac{1}{2}(a_n - ib_n) = c_n$, and $\frac{1}{2}(a_n + ib_n) = k_n$, we get from (1)

$$\text{(4)} \qquad f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + k_n e^{-inx}).$$

The coefficients $c_1, c_2, \cdots$, and $k_1, k_2, \cdots$ are obtained from (6b), (6c) in Sec. 11.1 and then (2) above with $t = nx$,

$$\text{(5)} \qquad \begin{aligned} c_n &= \frac{1}{2}\,(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i \sin nx)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx \\ k_n &= \frac{1}{2}\,(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i \sin nx)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\, dx. \end{aligned}$$

Finally, we can combine (5) into a single formula by the trick of writing $k_n = c_{-n}$. Then (4), (5), and $c_0 = a_0$ in (6a) of Sec. 11.1 give (summation from $-\infty$!)

$$\text{(6)} \qquad \begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx}, \\ c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx, \qquad n = 0, \pm 1, \pm 2, \cdots. \end{aligned}$$

This is the so-called *complex form of the Fourier series* or, more briefly, the **complex Fourier series,** of $f(x)$. The c_n are called the **complex Fourier coefficients** of $f(x)$.

For a function of period $2L$ our reasoning gives the **complex Fourier series**

$$\text{(7)} \qquad \begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\, dx, \qquad n = 0, \pm 1, \pm 2, \cdots. \end{aligned}$$

EXAMPLE 1 **Complex Fourier Series**

Find the complex Fourier series of $f(x) = e^x$ if $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ and obtain from it the usual Fourier series.

Solution. Since $\sin n\pi = 0$ for integer n, we have

$$e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = \cos n\pi = (-1)^n.$$

With this we obtain from (6) by integration

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\,dx = \frac{1}{2\pi}\frac{1}{1-in}\,e^{x-inx}\Big|_{x=-\pi}^{\pi} = \frac{1}{2\pi}\frac{1}{1-in}(e^{\pi} - e^{-\pi})(-1)^n.$$

On the right,

$$\frac{1}{1-in} = \frac{1+in}{(1-in)(1+in)} = \frac{1+in}{1+n^2} \qquad \text{and} \qquad e^{\pi} - e^{-\pi} = 2\sinh\pi.$$

Hence the complex Fourier series is

$$e^x = \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx} \qquad (-\pi < x < \pi). \tag{8}$$

From this let us derive the real Fourier series. Using (2) with $t = nx$ and $i^2 = -1$, we have in (8)

$$(1+in)e^{inx} = (1+in)(\cos nx + i\sin nx) = (\cos nx - n\sin nx) + i(n\cos nx + \sin nx).$$

Now (8) also has a corresponding term with $-n$ instead of n. Since $\cos(-nx) = \cos nx$ and $\sin(-nx) = -\sin nx$, we obtain in this term

$$(1-in)e^{-inx} = (1-in)(\cos nx - i\sin nx) = (\cos nx - n\sin nx) - i(n\cos nx + \sin nx).$$

If we add these two expressions, the imaginary parts cancel. Hence their sum is

$$2(\cos nx - n\sin nx), \qquad n = 1, 2, \cdots.$$

For $n = 0$ we get 1 (not 2) because there is only one term. Hence the real Fourier series is

$$e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{1+1^2}(\cos x - \sin x) + \frac{1}{1+2^2}(\cos 2x - 2\sin 2x) - + \cdots\right]. \tag{9}$$

In Fig. 270 the poor approximation near the jumps at $\pm\pi$ is a case of the Gibbs phenomenon (see CAS Experiment 20 in Problem Set 11.2). ■

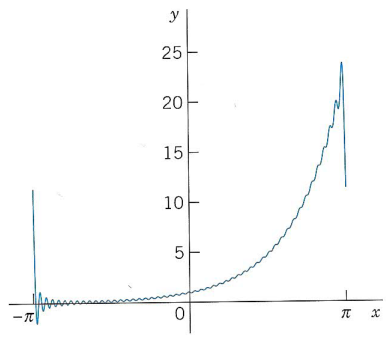

Fig. 270. Partial sum of (9), terms from $n = 0$ to 50