Fourier transform and PDEs

1. Fourier transform $\mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$

   $$\boxed{\mathcal{F}[e^{-ax^2}](w) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}}$$

   $$\boxed{\mathcal{F}[f * g](w) = \sqrt{2\pi}\, \mathcal{F}[f](w) \cdot \mathcal{F}[g](w)} \qquad f * g(x) = \int_{-\infty}^{\infty} f(y) g(x-y) dy$$

2. Partial differential equations (PDEs)

   Equations involving partial derivatives of the unknown

   Concepts: Order, linear, homogeneous, hyperbolic/parabolic/elliptic

   Solution: u solution of PDE in region R if

   (i) all derivatives appearing in PDE exist and are continuous in R

   (ii) u satisfy the PDE in all points in R

   Superposition/Linearity:

   u_1 and u_2 solve same *linear, homogeneous* PDE in R; $a, b \in \mathbb{R}$
   $\Rightarrow au_1 + bu_2$ solves same PDE in R

   Unique solution:

   Need also boundary and initial conditions!

Lecture 10: Partial differential equations (PDEs)

Kreyszig: Section 12.3, 12.4

1. Introduction to partial differential equations (continued)
2. Solution technique: Separation of variables
3. Example PDE: The wave equation

Homework:

1. Read Kreyszig 12.2 yourselves.
2. Repeat *Solution of ordinary differential equations* [Mat 3/Lin. Alg.]

$$
\begin{array}{lll}
(1) & u_{tt} = c^2 u_{xx} & t > 0,\ x \in (0, L) \\
(2) & u(0,t) = 0 = u(L,t) & t > 0,\ x \in \{0, L\} \\
(3) & u(x,0) = f(x) & t = 0,\ x \in (0, L) \\
(4) & u_t(x,0) = g(x) & t = 0,\ x \in (0, L)
\end{array}
$$

1. Product solutions $u(x,t) = F(x)G(t)$

(a) Reduction to ODEs, (1) and (2):

$$F'' - kF = 0$$

$$G'' - c^2 kG = 0 \qquad (k = \text{constant})$$

$$F(0) = 0 = F(L)$$

Lecture 10: Separation of variables

$$
\begin{array}{lll}
(1) & u_{tt} = c^2 u_{xx} & t > 0,\ x \in (0, L) \\
(2) & u(0,t) = 0 = u(L,t) & t > 0,\ x \in \{0, L\} \\
(3) & u(x,0) = f(x) & t = 0,\ x \in (0, L) \\
(4) & u_t(x,0) = g(x) & t = 0,\ x \in (0, L)
\end{array}
$$

1. Product solutions $u(x,t) = F(x)G(t)$

(b) Solve for F: $F_n(x) = \sin \frac{n\pi x}{L}$

(c) Solve for G: $G_n(t) = B_n \cos \frac{cn\pi t}{L} + B_n^* \sin \frac{cn\pi t}{L}$

(d) All product solution:

$$u_n(x,t) = F_n(x)G_n(t),\ n \in \{0, 1, 2, 3, \dots\}$$

2. Superposition and (3) and (4)

$$u(x,t) = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \left(B_n \cos \frac{cn\pi t}{L} + B_n^* \sin \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L}$$

Nonhomogeneous initial conditions:

$$f(x) \overset{(3)}{=} u(x,0) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi x}{L}$$

$$\underset{F-sin\ \ series}{\Rightarrow} \boxed{B_n = \frac{2}{L} \int_0^L f \sin \frac{n\pi x}{L} dx}$$

$$g(x) \overset{(4)}{=} u_t(x,0) = \sum_{n=0}^{\infty} B_n^* \frac{cn\pi}{L} \sin \frac{n\pi x}{L}$$

$$\Rightarrow \boxed{B_n^* \frac{cn\pi}{L} = \frac{2}{L} \int_0^L g \sin \frac{n\pi x}{L} dx}$$

Separation of variables – wave equation

(1)	$u_{tt} = c^2 u_{xx}$	$t > 0,\ x \in (0, L)$
(2)	$u(0,t) = 0 = u(L,t)$	$t > 0,\ x \in \{0, L\}$
(3)	$u(x,0) = f(x)$	$t = 0,\ x \in (0, L)$
(4)	$u_t(x,0) = g(x)$	$t = 0,\ x \in (0, L)$

1. Separation of variables $u(x,t) = F(x)G(t)$

   (1) and (2) $\Rightarrow$ $\boxed{F'' - kF = 0}$ $\boxed{G'' - c^2 kG = 0}$ $\boxed{F(0) = 0 = F(L)}$

2. Find *all* $u = FG$ solutions of (1) and (2) [linear, homogeneous]

   Only $u \neq 0$ if $k = -(\frac{n\pi}{L})^2$

   $\boxed{F_n(x) = \sin \frac{n\pi x}{L}}$ $\boxed{G_n(t) = B_n \cos \frac{cn\pi t}{L} + B_n^* \sin \frac{cn\pi t}{L}}$

   $\boxed{u_n(x,t) = F_n(x)G_n(t)}$ $n \in \{0, 1, 2, 3, \dots\}$

Separation of variables – wave equation

$$
\begin{array}{llll}
(1) & u_{tt} = c^2 u_{xx} & & t > 0,\ x \in (0, L) \\
(2) & u(0,t) = 0 = u(L,t) & & t > 0,\ x \in \{0, L\} \\
(3) & u(x,0) = f(x) & & t = 0,\ x \in (0, L) \\
(4) & u_t(x,0) = g(x) & & t = 0,\ x \in (0, L)
\end{array}
$$

4. Superposition and (3) and (4) [inhomogeneous cond'ns]

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \left(B_n \cos \frac{cn\pi t}{L} + B_n^* \sin \frac{cn\pi t}{L}\right) \sin \frac{n\pi x}{L}}$$

$$f(x) \overset{(3)}{=} u(x,0) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi x}{L} \underset{F-\sin\ series}{\Rightarrow} \boxed{B_n = \frac{2}{L} \int_0^L f \sin \frac{n\pi x}{L} dx}$$

$$g(x) \overset{(4)}{=} u_t(x,0) = \sum_{n=0}^{\infty} B_n^* \frac{cn\pi}{L} \sin \frac{n\pi x}{L} \Rightarrow \boxed{B_n^* \frac{cn\pi}{L} = \frac{2}{L} \int_0^L g \sin \frac{n\pi x}{L} dx}$$

u, u_t sinus series at $t = 0$ $\Rightarrow$ use Fourier sinus series of f, g

u solution if series converges and 2x term-wise differentiation ok