

Summary: Heat and Laplace equation

1 Boundary value problems:

Cauchy u given at $t = 0$.

Dirichlet u given on boundary

Neumann (normal) derivative of u given on boundary

2 Heat equation:

$$(1) \quad u_t = c^2 u_{xx} \quad t > 0, x \in (0, L)$$

$$(2) \quad u(0, t) = 0 = u(L, t) \quad t > 0, x \in \{0, L\}$$

$$(2') \quad u_x(0, t) = 0 = u_x(L, t) \quad t > 0, x \in \{0, L\}$$

$$(3) \quad u(x, 0) = f(x) \quad t = 0, x \in (0, L)$$

u temperature of rod, ends: fixed temperature (2) or insulated (2')

Cauchy-Dirichlet (1), (2), (3); Cauchy-Neumann (1), (2'), (3)

3 Laplace equation: $u_{xx} + u_{yy} = 0$

Electrostatic potential, potential flow, membrane, temperature ...

4 Solved by separation of variables, $u = F(x)G(t)$...

Review: Fourier transform

$$\textcircled{1} \quad \mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx} dx$$

$$\textcircled{2} \quad \mathcal{F}[f'](w) = (iw)\mathcal{F}[f](w)$$

$$\textcircled{3} \quad \mathcal{F}[e^{-iax}f(x)](w) = \mathcal{F}[f](w + a)$$

$$\textcircled{4} \quad \mathcal{F}^{-1}[e^{iaw}\hat{f}(w)](x) = f(x + a)$$

$$\textcircled{5} \quad \mathcal{F}[f * g](w) = \sqrt{2\pi} \mathcal{F}[f](w) \cdot \mathcal{F}[g](w)$$

$$\text{where } (f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$$

$$\textcircled{6} \quad \mathcal{F}[e^{-ax^2}](w) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$$

Lecture 12: Partial differential equations

Kreyszig: Section 12.4, 12.5, 12.7

1 PDEs:

Heat equation (with derivation)

Wave equation

2 Solution techniques:

Fourier transform

Method of characteristics (D'Alembert)

Lecture 12: The heat equation in IR

$$(1) \quad \begin{cases} u_t = c^2 u_{xx} & t > 0, x \in \mathbb{R} \\ u(x, 0) = f(x) & t = 0, x \in \mathbb{R} \end{cases}$$

1. Derivation

2. Solution using Fourier transform

(a) Fourier transform problem

$$\hat{u}_t = -c^2 w^2 \hat{u}, \quad \hat{u}(x, 0) = \hat{f}(x)$$

(b) Solve

$$\hat{u}(w, t) = \hat{f}(w) e^{-c^2 w^2 t}$$

(c) Inverse Fourier transform

$$u(x, t) = (f * g)(x), \quad g = \mathcal{F}^{-1}[e^{-c^2 w^2 t}]$$

Lecture 12: The wave equation in \mathbb{R}

$$(3) \quad u_{tt} = c^2 u_{xx} \quad t > 0, x \in \mathbb{R}$$

$$(6) \quad u(x, 0) = f(x) \quad t = 0, x \in \mathbb{R}$$

$$(7) \quad u_t(x, 0) = g(x) \quad t = 0, x \in \mathbb{R}$$

1. Solution by Fourier transform

2. D'Alembert's solution

$$y = x + ct, \quad s = x - ct, \quad v(y, s) = u(x, t)$$

$$\implies v_{ys} = 0 \implies \underset{\text{integrate}}{v(y, s) = \phi(y) + \psi(s)}$$

$$\implies \boxed{u(x, t) = \phi(x + ct) + \psi(x - ct)}$$

Partial differential equations

- 1 Heat equation: $u_t = c^2 u_{xx}$

Derived from Fourier's law and conservation of energy

- 2 Solution by Fourier transform:

$$u_t = c^2 u_{xx}, \quad u(x, 0) = f(x)$$

$$\xrightarrow{\mathcal{F}} \hat{u}_t = -c^2 w^2 \hat{u}, \quad \hat{u}(w, 0) = \hat{f}(w)$$

$$\xrightarrow{\text{solve}} \hat{u}(w, t) = \hat{f}(w) e^{-c^2 w^2 t}$$

$$\xrightarrow{\mathcal{F}^{-1}} u(x, t) = (f * g)(x, t) = \int_{-\infty}^{\infty} f(y) g(x - y) dy,$$

$$\text{where } g(x) = \mathcal{F}^{-1}[e^{-c^2 w^2 t}](x) = \frac{1}{\sqrt{4\pi c^2 t}} e^{-\frac{x^2}{4c^2 t}}$$

- 3 D'Alembert's solution of wave equation

$$u_{tt} = c^2 u_{xx} \quad \implies \quad \begin{array}{l} y = x + ct, \\ s = x - ct, \\ v(y, s) = u(x, t) \end{array} \quad \implies \quad v_{ys} = 0 \quad \xrightarrow{\text{integrate}} \quad v(y, s) = \phi(y) + \psi(s)$$

$$u(x, t) = v(x + ct, x - ct) = \phi(x + ct) + \psi(x - ct)$$