

Summary Lecture 18: Complex line integrals

1 Complex line integral:

$$\int_C [af(z) + bg(z)]dz = a \int_C f(z)dz + b \int_C g(z)dz$$

$$\int_{-C} f(z)dz = - \int_C f(z)dz$$

$$\int_{C_1 \cup C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \quad \text{when } C_1 \cap C_2 = \emptyset$$

$$\boxed{|\int_C f(z)dz| \leq M \cdot L} \quad M = \max_{z \in C} |f(z)|, \quad L = \text{length of } C$$

2 Cauchy's integral theorem

f analytic in simply connected domain D , $C \subset D$ simple, closed curve

$$\Rightarrow \oint_C f(z)dz = 0$$

3 Consequences:

(a) $\int_C f(z)dz$ is independent of path in D

(b) The indefinite integral of f exists in D , i.e. a function F s.t.

$$F'(z) = f(z) \quad \text{and} \quad \int_{z_1}^{z_2} f(z) dz = F(z_2) - F(z_1).$$

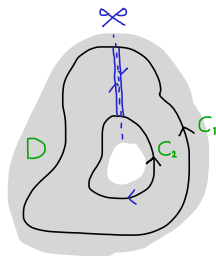
Summary Lecture 18: Domains with holes

4 Domains with holes:

Cut to have a simply connected domain...

... add segments along cut to have closed curve...

Then use Cauchy:



Cauchy in the cut domain D^* and cancelations along cut:

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$

Summary Lecture 18: Example

C_1 any simple, closed curve surrounding $z = z_0$

C_1 : circle $|z - z_0|^2 = r^2$

Cauchy's integral theorem in domain with one hole, $m \in \mathbb{Z}$:

$$\oint_{C_1} (z - z_0)^m dz = \oint_{C_2: z(t)=z_0+re^{it}} (z - z_0)^m dz$$

$$\underset{\substack{\text{last time} \\ \text{Eks.1}}}{=} \begin{cases} 2\pi i, & m = -1, \\ 0, & m \neq -1, m \in \mathbb{Z} \end{cases}$$

Lecture 19: Complex Analysis

Kreyszig: Sections 14.3, 14.4

- 1 Cauchy integral formula
- 2 Analytic functions infinitely differentiable
- 3 Properties of analytic functions:
Cauchy's inequality and Liouville's theorem
- 4 Examples

Lecture 19: Cauchy's integral formula

(A1) f is analytic in simply connected domain D

(A2) $z_0 \in D$, $C \subset D$ simple closed curve,
positively oriented, enclosing z_0

Then
$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

Lecture 19: Analytic \Rightarrow infinitely differentiable

f analytic in $D \Rightarrow f$ ∞ -differentiable in D , and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Lecture 19: Properties of analytic functions

Cauchy's inequality:

$$f \text{ analytic} \implies |f^{(n)}(z_0)| = \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|$$

Gauss mean value theorem:

$$f \text{ analytic} \implies f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt$$

Liouville's theorem:

$$f \text{ analytic, bounded in } \mathbb{C} \implies f \text{ is constant}$$

Lecture 19: Properties of analytic functions

Cauchy's inequality:

$$f \text{ analytic in } |z - z_0| \leq r \implies |f^{(n)}(z_0)| = \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|$$

Gauss mean value theorem:

$$f \text{ analytic in } |z - z_0| \leq r \implies f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt$$

Liouville's theorem:

$$f \text{ analytic, bounded in } \mathbb{C} \implies f \text{ is constant}$$

Maximum (modulus) principle:

$$f \text{ analytic in domain } D \text{ and } |f(z)| \text{ attains its max in } D \\ \implies f \text{ is constant in } D$$

Morera's theorem:

$$f \text{ continuous in } D \text{ and } \oint_C f(z) dz = 0 \text{ for all simple, closed } C \subset D \\ \implies f \text{ analytic in } D$$

Summary Lecture 19

- 1 Cauchy's integral formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \quad \text{if}$$

(A1) f is analytic in simply connected domain D

(A2) $z_0 \in D$, $C \subset D$ simple closed curve, positively oriented, enclosing z_0 .

- 2 Infinitely differentiable:

f analytic in $D \Rightarrow f$ infinitely differentiable in D , and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

- 3 Properties of analytic functions:

Cauchy's inequality: $|f^{(n)}(z_0)| = \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|$ if f analytic

Liouville's theorem: f analytic, bounded in $\mathbb{C} \Rightarrow f$ is constant

Morera's theorem: f continuous in D and $\oint_C f(z) dz = 0$ for all simple, closed $C \subset D \Rightarrow f$ analytic in D