Summary Lecture 19: Cauchy's integral theorem

1. Cauchy's integral formula: $\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz}$ if

   (A1) f is analytic in simply connected domain D

   (A2) $z_0 \in D$, $C \subset D$ simple closed curve, positively oriented, enclosing z_0.

2. Infinitly differentiable:

   f analytic in D $\Rightarrow$ f infinitely differentiable in D, and

   $$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz}$$

3. Properties of analytic functions:

   Cauchy's inequality: $\boxed{|f^{(n)}(z_0)| = \frac{n!}{r^n} \max_{|z - z_0| = r} |f(z)|}$ if f analytic

   Liouville's theorem: f analytic, bounded in $\mathbb{C}$ $\Rightarrow$ f is constant

   Morera's theorem: f continuous in simply connected domain D and $\oint_C f(z)dz = 0$ for all simple, closed $C \subset D$ $\Rightarrow$ f analytic in D

Lecture 20: Complex power series

Kreyszig: Sections 15.1, 15.2

1. Complex sequences and series
2. Complex power series
3. Convergence and divergence
4. Radius of convergence
5. Examples

Homework:

Repeat from Mat 1/GKA 2:

Taylor series. How to find/work with them, Taylor's thm, examples

Lecture 20: Complex series and sequences

Convergence (absolute/not), divergence, Cauchy

Geometric series

Convergence tests:

Comparison, ratio, root and divergence tests

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

Center z_0, coefficients a_n $\quad [(z-z_0)^0 = 1]$

Convergence in z_1

$\Rightarrow$ convergence in z for all $|z-z_0| < |z_1 - z_0|$

Divergence in z_2

$\Rightarrow$ divergence in z for all $|z-z_0| > |z_2 - z_0|$

Distance $R = |z_0 - z^*|$ to nearest point z^* where power series diverges

- Exists always

- Series converges (diverges) if $|z - z_0| < R$ $(> R)$

Cauchy-Hadamard:

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \quad \text{(when the limit exists)}$$

Summary Lecture 20: Complex power series

1. Complex series and sequences:

   Definitions, results and proofs – similar to real case

   Convergence, absolute convergence, divergence

   Comparison, ratio, and root test; divergence test; geometric series

2. Complex power series:

   $$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

   Center z_0, coefficients a_n, $(z-z_0)^0 = 1$ by definition

   Convergence in z_1 $\Rightarrow$ convergence in z for all $|z-z_0| < |z_1-z_0|$

   Divergence in z_2 $\Rightarrow$ divergence in z for all $|z-z_0| > |z_2-z_0|$

3. Radius of convergence R:

   Distance $R = |z_0 - z^*|$ to nearest point z^* where power series diverges

   Always exists; series converges (diverges) if $|z-z_0| < R$ ($> R$)

   Cauchy-Hadamard: $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ when the limit exists