Summary Lecture 20: Complex (power) series

1. Complex series and sequences:

   Definitions, results and proofs – similar to real case

   Convergence, absolute convergence, divergence

   Comparison, ratio, and root test; divergence test; geometric series

2. Complex power series:

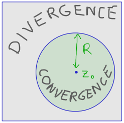
   $$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

   Center z_0, coefficients a_n, $(z-z_0)^0 = 1$ by definition

   Convergence in z_1 $\Rightarrow$ convergence in z for all $|z-z_0| < |z_1 - z_0|$

   Divergence in z_2 $\Rightarrow$ divergence in z for all $|z-z_0| > |z_2 - z_0|$

3. Radius of convergence R:



   Distance $R = |z_0 - z^*|$ to nearest point z^* where power series diverges

   Always exists; series converges (diverges) if $|z-z_0| < R$ ($> R$)

   Cauchy-Hadamard: $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ when the limit exists

Lecture 21: Complex power series

Kreyszig: Sections 15.3, 15.4

1. Term wise operations on power series
2. Power series representation of functions
3. Taylor's formula, Taylor series, definition and convergence
4. Examples

$$c_1 \sum_{n=0}^{\infty} a_n z^n + c_2 \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n) z^n \qquad [R = \min(R_a, R_b)]$$

$$\left(\sum_{n=0}^{\infty} a_n z^n \right)' = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad [R = R_a]$$

$$\int \left(\sum_{n=0}^{\infty} a_n z^n \right) dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} + C \qquad [R = R_a]$$

Represents their sum where they converges

Uniqueness:

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n \quad \Rightarrow \quad a_n = b_n, n = 0, 1, 2, \ldots$$

The sum is analytic where the power series converges

Taylor series: $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

Taylor's formula: $$f(z) = \sum_{n=0}^{m} a_n (z - z_0)^n + R_m(z, z_0)$$

Taylor's theorem:

f analytic in domain D and $z_0 \in D$

$\Longrightarrow$ the Taylor series of f about z_0 exists, is unique, and converges

to f in the largest open disc about z_0 where f is analytic

Summary Lecture 21: Complex power series

1. Termwise operations on power series:

   a. $c_1 \sum_{n=0}^{\infty} a_n z^n + c_2 \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n) z^n \quad [R = \min(R_a, R_b)]$

   b. $\left(\sum_{n=0}^{\infty} a_n z^n \right)' = \sum_{n=1}^{\infty} n a_n z^{n-1} \quad [R = R_a]$

   c. $\int \left(\sum_{n=0}^{\infty} a_n z^n \right) dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} + C \quad [R = R_a]$

2. Representation of functions by power series:

   Represents their sum where they converges

   Uniqueness: $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n \quad \Rightarrow \quad a_n = b_n, n = 0, 1, 2, \dots$

   The function is analytic where the power series converges

Summary Lecture 21: Complex power series

3. Taylor series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

4. Taylor's theorem:

   If f is analytic in a domain D and $z_0 \in D$, then

   (a) the Taylor series of f about z_0 exists and is unique,

   (b) and it converges and equals f in the largest open disc about z_0 where f is analytic