Summary Lecture 23: Singularities

1. Singularity: Point z_0 where $f(z)$ is not analytic/defined $(\ldots)$

2. Isolated singularity:

   z_0 is the only singularity in $|z - z_0| < R$ for some $R > 0$

   $\Downarrow$ **Laurent's theorem**

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\neq 0}, \quad 0 < |z - z_0| < R$$

Principal value of z_0: $\boxed{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}$

z_0 order n pole: $b_n \neq 0,\ b_k = 0,\ k > n$

z_0 isolated essential singularity: $b_n \neq 0$ for ∞ many n

Summary Lecture 23: Residue integration

3. Residue integration:

   Goal: $\oint_C f(z)dz = 2\pi i \sum \text{residues}$

4. Only one singularity:

   z_0 only singularity of $f(z)$ enclosed by C (simple, closed, counter cl.wise)

   $\Downarrow$ **Laurent's theorem**

   $$f(z) = a_0 + a_1(z - z_0) + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots, \quad 0 < |z - z_0| < R$$

   $\Downarrow$ **Term wise integration/Laurent's theorem**

   $$\oint_C f(z)dz = 2\pi i b_1$$

5. Residue of $f(z)$ at z_0: $\operatorname*{Res}_{z=z_0} f(z) = b_1$

   b_1-coefficient in Laurent series that converges in $0 < |z - z_0| < R$!

Lecture 24: Residue integration

Kreyszig: Sections 16.3, 16.4

1. Residue integration
2. Formulas for residue of a pole
3. Residue theorem
4. Application: Computing real integrals
5. Examples

Frist øving 12 og 13:

Neste uke - onsdag kl 23:59!

Info om eksamen, øvinger, treffetid:

Se www.

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 \text{ where}$$

$$f(z) = \sum a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots, \quad 0 < |z-z_0| < R$$

z_0 order 1 pole:

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0) f(z) \quad \text{or} \quad \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

z_0 order n pole:

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z\to z_0} \left((z-z_0)^n f(z) \right)^{(n-1)}$$

z_0 isolated essential singularity:

Find b_1 from the Laurent series!

Assume

(A1) C simple, closed curve oriented counterclockwise

(A2) $f(z)$ has finite number of singularities $z_1, \ldots, z_m$ enclosed by C

Then

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} f(z).$$

Summary Lecture 24: Residue integration

1. Residues: $\operatorname*{Res}_{z=z_0} f(z) = b_1$ where

$$f(z) = \sum a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots,\ 0 < |z-z_0| < R$$

z_0 order 1 pole:

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0) f(z)} \quad \text{or} \quad \boxed{\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}}$$

z_0 order n pole:

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z\to z_0} \Big((z-z_0)^n f(z) \Big)^{(n-1)}}$$

z_0 isolated essential singularity:
Find b_1 from the Laurent series!

2. Residue theorem: Assume
   (A1) C simple, closed curve oriented counterclockwisely
   (A2) $f(z)$ has finite number of singularities $z_1, \ldots, z_m$ enclosed by C

Then $$\boxed{\oint_C f(z) dz = 2\pi i \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} f(z)}$$