

Summary: Laplace Transform

1 $F(s) = \mathcal{L}[f](s) = \int_0^{\infty} f(t)e^{-st} dt$

2 Unit step function:

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases} \quad \mathcal{L}[u(t-a)](s) = \frac{1}{s} e^{-as}$$

3 Properties:

Linearity: $\mathcal{L}[af(t) + bg(t)](s) = a\mathcal{L}[f](s) + b\mathcal{L}[g](s)$

s-shift: $\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a)$ for $s-a > k$

t-shift:

$$\mathcal{L}[f(t-a)u(t-a)](s) = e^{-as}\mathcal{L}[f](s)$$

Derivatives: $\mathcal{L}[f'(t)](s) = s\mathcal{L}[f](s) - f(0)$

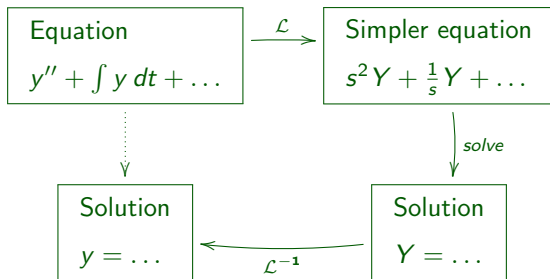
$$\mathcal{L}[f''(t)](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0)$$

Integral:

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right](s) = \frac{1}{s}\mathcal{L}[f](s)$$

Summary: Laplace Transform

- 5 Solving equations with the Laplace transform:



Lecture 3: Laplace Transform

Kreyszig: Sections 6.4 and 6.5

- 1 δ -functions
- 2 Convolutions
- 3 Integral representation of solutions of ODEs
- 4 On finding partial fractions
- 5 Many examples

Lecture 3: Laplace Transform

Delta-function:

$$\delta(t - a) = \lim_{h \rightarrow 0} \frac{u(t - a) - u(t - (a + h))}{h} \quad ,,$$

Laplace-transform:

$$\mathcal{L}[\delta(t - a)](s) = e^{-as}$$

Lecture 3: Laplace Transform

Convolution:

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau$$

Laplace transform:

$$\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)$$

Lecture 3: Laplace Transform

$$y'' + ay' + by = r(t), \quad t > 0,$$
$$y(0) = 0 = y'(0).$$

Integral representation:

$$y(t) = (q * r)(t) = \int_0^t q(t - \tau)r(\tau) d\tau$$

where

$$q(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + as + b}\right](t)$$

Partial fraction decomposition

$P(s)$ and $Q(s)$ polynomials, no common factor, $\text{order}(P) < \text{order}(Q)$

① $Q(s) = (s - s_1)(s - s_2)(s - s_3) \dots$ non-repeated factors

$$\Rightarrow \frac{P(s)}{Q(s)} = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \frac{A_3}{s - s_3} + \dots$$

② $Q(s) = (s - s_0)^n \dots$ repeated factors

$$\Rightarrow \frac{P(s)}{Q(s)} = \frac{A_1}{s - s_0} + \frac{A_2}{(s - s_0)^2} + \dots + \frac{A_n}{(s - s_0)^n} + \dots$$

③ $Q(s) = (s^2 + b_1s + a_1)(s^2 + b_2s + a_2) \dots$ irreducible, non-repeated quad. factors

$$\Rightarrow \frac{P(s)}{Q(s)} = \frac{A_1s + B_1}{s^2 + b_1s + a_1} + \frac{A_2s + B_2}{s^2 + b_2s + a_2} + \dots$$

④ $Q(s) = (s^2 + b_1s + a_1)^n \dots$ see earlier mathematics courses.

Partial fraction decomposition

Eksample 1:

$$\frac{s^5 + 2}{s(s+1)(s-1)^2(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{Es + F}{s^2+1}$$

Eksample 2:

$$\frac{1}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

Multiply by denominator $(s^2+1)(s^2+2s+2)$

$$1 = 1 + 0s + 0s^2 + 0s^3$$

$$= (As+B)(s^2+2s+2) + (Cs+D)(s^2+1)$$

$$= (2B+D) + (2B+2A+C)s + (2A+B+D)s^2 + (A+C)s^3$$

Coefficients of same powers of s must coincide:

$$\begin{cases} 1 = 2B + D \\ 0 = 2B + 2A + C \\ 0 = 2A + B + D \\ 0 = A + C \end{cases} \Rightarrow \begin{cases} A = -\frac{2}{5} \\ B = \frac{1}{5} \\ C = \frac{2}{5} \\ D = \frac{3}{5} \end{cases}$$

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② **Properties:**

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s-shift: $\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s - a)$ for $s - a > k$

t-shift: $\mathcal{L}[f(t - a)u(t - a)](s) = e^{-as}\mathcal{L}[f](s)$

Derivatives: $\mathcal{L}[f'(t)](s) = s\mathcal{L}[f](s) - f(0)$

Integral: $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right](s) = \frac{1}{s}\mathcal{L}[f](s)$

Convolution: $\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)$ $(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau$

③ **Heaviside, delta functions:**

δ -function: $\int_0^{\infty} \delta(t - a)f(t)dt = f(a)$ for all continuous f .

$\mathcal{L}[u(t - a)] = \frac{1}{s}e^{-as}$, $\mathcal{L}[\delta(t - a)] = e^{-as}$