# Fourier Series

1. $f(x)$ is $p$-periodic if $f(x+p) = f(x)$ for all $x \in \mathbb{R}$.

2. Fourier Series:

   Representation of periodic functions by trigonometric series,

   $$(1) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} \Big( a_n \cos nx + b_n \sin nx \Big) \qquad (p = 2\pi).$$

3. Questions:

   What is $a_n$ and $b_n$?

   When does the series (1) converges?

   When and where is its sum equal $f(x)$?

4. Comments

   Fourier series can represent discontinuous functions!!

   Fourier series are a very important tool in science and technology.

# Lecture 5: Fourier Series

Kreyszig: Section 11.1

1. Periodic functions
2. The trigonometric system
3. Fourier series, the coefficients
4. Fourier series, convergence and sum
5. Examples

$p$-periodic functions:

$$f(x) = f(x+p) \quad \text{for all} \quad x \in \mathbb{R}.$$

Trigonometric system:

$$\mathcal{T} = \{1, \cos x, \sin x, \ldots, \cos nx, \sin nx, \ldots\}$$

($2\pi$-periodic)

Orthogonality:

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x)dx = 0$$

for $f, g \in \mathcal{T}$, $f \neq g$.

# Lecture 5: Fourier Series

Fourier series for $2\pi$-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx,$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx$$

## Convergence and sum

Assume

(A1) $f$ periodic and piecewise continuous.

(A2) $f$ has both right and left derivatives at $x$.

Then the Fourier series $S_f$ converge at $x$ and

$$S_f(x) = \frac{f(x^+) + f(x^-)}{2},$$

i.e. $S_f(x) = f(x)$ if $f$ is also continuous at $x$.

# Summary: Fourier series

1. Periodic functions:
   p-periodic: $f(x) = f(x+p) \ \ (= f(x+np),\ n \in \mathbb{N}), \quad x \in \mathbb{R}.$

   $2\pi$-periodic: $a_0 + \sum_{n=1}^{\infty} \Big( a_n \cos nx + b_n \sin nx \Big)$

2. Trigonometric system:
   $\mathcal{T} = \{1, \cos x, \sin x, \dots, \cos nx, \sin nx, \dots\}$ ($2\pi$-periodic)

   Orthogonality:

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x)dx = 0 \quad \text{for} \quad f, g \in \mathcal{T},\ f \neq g.$$

3. Fourier series for $2\pi$-periodic $f(x)$:

   $S_f(x) = a_0 + \sum_{n=1}^{\infty} \Big( a_n \cos nx + b_n \sin nx \Big)$ where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \ dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \ dx$$

# Summary: Fourier series

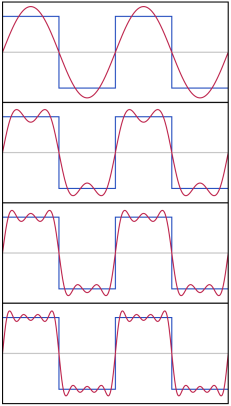

Figure: From Wikipedia

5. Convergence and sum

Assume

(A1) $f$ periodic and piecewise continuous.

(A2) $f$ has both right and left derivatives at $x$.

Then the Fourier series $S_f$ converge at $x$ and

$$S_f(x) = \frac{f(x^+) + f(x^-)}{2},$$

i.e. $S_f(x) = f(x)$ if $f$ is also continuous at $x$.

`Animation by Yuya` [a]

[a] Oscillations near discontinuities do not die out – Gibb's phenomenon!