Fourier series

1. Periodic functions:

   p-periodic: $f(x) = f(x+p) \;\; (= f(x+np),\; n \in \mathbb{N}), \quad x \in \mathbb{R}$.

   2π-periodic: $a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big)$

2. Trigonometric system:

   $\mathcal{T} = \{1, \cos x, \sin x, \ldots, \cos nx, \sin nx, \ldots\}$ (2π-periodic)

   Orthogonality:

   $$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x) g(x) dx = 0 \quad \text{for} \quad f, g \in \mathcal{T},\; f \neq g.$$

3. Fourier series for 2π-periodic $f(x)$:

   $S_f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big)$ where

   $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,$$

   $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx,$$

   $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx$$

Fourier series

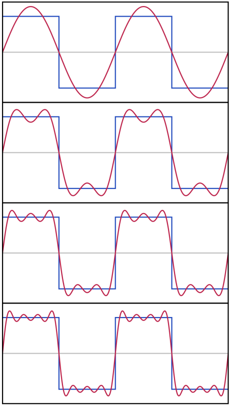

Figure: From Wikipedia

5. Convergence and sum

Assume

(A1) f periodic and piecewise continuous.

(A2) f has both right and left derivatives at x.

Then the Fourier series S_f converge at x and

$$S_f(x) = \frac{f(x^+) + f(x^-)}{2},$$

i.e. $S_f(x) = f(x)$ if f is also continuous at x.

`Animation by Yuya` [a]

[a] Oscillations near discontinuities do not die out – Gibb's phenomenon!

Lecture 6: Fourier Series

Kreyszig: Section 11.2

1. Fourier series with periode $2L$
2. Odd and even functions
3. Fourier sin and cos series
4. Odd and even periodic extensions
5. Examples

Homework: Read yourselves Kreyszig section 11.3.

Revise before next week: Complex numbers, $e^{ix} = \cos x + i \sin x$.

Fourier series of $p = 2L$-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \text{ where}$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

Odd and even functions

Even $g(-x) = g(x)$

$$\boxed{\int_{-L}^{L} g(x)dx = 2\int_0^L g(x)dx}$$

Odd $h(-x) = -h(x)$

$$\boxed{\int_{-L}^{L} h(x)dx = 0}$$

even $\cdot$ even $=$ odd $\cdot$ odd $=$ even; odd $\cdot$ even $=$ odd

Fourier cos series:

f even: $S_f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$,

$$a_0 = \frac{1}{L}\int_0^L f(x)dx,\ a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

Fourier sin series:

f odd: $S_f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Even $2L$-periodic extensions of $f(x)$, $0 \leq x \leq L$

$f_1(x)$, $x \in \mathbb{R}$, even, $2L$-periodic, $f_1 = f$ on $[0, L]$

$S_{f_1}(x) =$ cos-series $=:$ the Fourier cos series of f

Odd $2L$-periodic extensions of $f(x)$, $0 \leq x \leq L$

$f_2(x)$, $x \in \mathbb{R}$, odd, $2L$-periodic, $f_2 = f$ on $[0, L]$

$S_{f_2}(x) =$ sin-series $=:$ the Fourier sin series of f

Summary: Fourier series

1. Fourier series of $p = 2L$-periodic $f(x)$:

   $S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$ where

   $$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx,$$

   $$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx,$$

   $$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

2. Odd and even functions

   Even $g(-x) = g(x)$ $\quad \boxed{\int_{-L}^{L} g(x)dx = 2\int_0^L g(x)dx}$

   Odd $h(-x) = -h(x)$ $\quad \boxed{\int_{-L}^{L} h(x)dx = 0}$

   even $\cdot$ even $=$ odd $\cdot$ odd $=$ even; odd $\cdot$ even $=$ odd

Summary: Fourier series

4. Fourier sin and cos series:

   f even: $S_f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$,

   $$\boxed{a_0 = \frac{1}{L}\int_0^L f(x)dx}, \quad \boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx}$$

   f odd: $S_f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$,

   $$\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx}$$

5. Even and odd $2L$-periodic extensions:

   $f(x)$, $0 \leq x \leq L$

   $f_1(x)$, $x \in \mathbb{R}$, even, $2L$-periodic, $f_1 = f$ on $[0, L]$

   $S_{f_1}(x) =$ cos-series $=:$ the Fourier cos series of f

   $f_2(x)$, $x \in \mathbb{R}$, odd, $2L$-periodic, $f_2 = f$ on $[0, L]$

   $S_{f_2}(x) =$ sin-series $=:$ the Fourier sin series of f