

Summary: Fourier series

- 1 Fourier series of $p = 2L$ -periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \text{ where}$$

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx,$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx,$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

- 2 Odd and even functions

Even $g(-x) = g(x)$

$$\int_{-L}^L g(x) dx = 2 \int_0^L g(x) dx$$

Odd $h(-x) = -h(x)$

$$\int_{-L}^L h(x) dx = 0$$

even \cdot even = odd \cdot odd = even; odd \cdot even = odd

Summary: Fourier series

4 Fourier sin and cos series:

$$f \text{ even: } S_f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

$$a_0 = \frac{1}{L} \int_0^L f(x) dx, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

$$f \text{ odd: } S_f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

5 Even and odd $2L$ -periodic extensions of $f(x)$, $0 \leq x \leq L$:

$$f_1(x), x \in \mathbb{R}, \text{ even, } 2L\text{-periodic, } f_1 = f \text{ on } [0, L],$$

$$S_{f_1}(x) = \text{cos-series} =: \text{the Fourier cos series of } f$$

$$f_2(x), x \in \mathbb{R}, \text{ odd, } 2L\text{-periodic, } f_2 = f \text{ on } [0, L],$$

$$S_{f_2}(x) = \text{sin-series} =: \text{the Fourier sin series of } f$$

6 **OBS:** $f = f_1 = f_2$ on $[0, L]$, $f_1 \neq f_2$, f not defined on $[0, L]^c$!

Lecture 7: Fourier Series

Kreyszig: Section 11.4 (10th ed.) and Section 11.4 in 9th ed.!

- 1 Approximation with trigonometric polynomials
- 2 Bessel's inequality, Parseval's identity
- 3 Complex Fourier series
- 4 Examples

Section 11.4 in 9th ed. available on course wiki page (Fremdriftsplan).

Homework: Repeat complex *numbers*, *absolute values*, *exponentials*
[Mat 3/Lin. Alg.]

Lecture 7: Fourier Series

Approximation of $f(x)$ by trigonometric polynomial

$$P_k(x) = A_0 + \sum_{n=1}^k (A_n \cos nx + B_n \sin nx)$$

Least mean square (L^2) error when $P_k = S_{f,k}$:

$$\int_{-\pi}^{\pi} |f(x) - S_{f,k}(x)|^2 dx \leq \int_{-\pi}^{\pi} |f(x) - P_k(x)|^2 dx$$

for all $P_k(x)$

where the Fourier k -th partial sum

$$S_{f,k}(x) = a_0 + \sum_{n=1}^k (a_n \cos nx + b_n \sin nx)$$

Lecture 7: Fourier Series

$$\|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^k (a_n^2 + b_n^2) \right]$$

Bessel's inequality

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx$$

Parseval's identity

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx$$

Lecture 7: Fourier Series

Fourier series of $2L$ -periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

Complex Fourier series of $2L$ -periodic $f(x)$:

$$S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}}, \quad c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-i \frac{n\pi x}{L}} dx$$

Summary: Fourier series

- 1 **Fourier series** of 2π -periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$S_{f,k}(x) = a_0 + \sum_{n=1}^k (a_n \cos nx + b_n \sin nx) \quad k\text{-th partial sum}$$

- 2 **Approximation of $f(x)$ by trigonometric polynomial**

$$P_k(x) = A_0 + \sum_{n=1}^k (A_n \cos nx + B_n \sin nx)$$

Mean square (or L^2) error:

$$\|f - P_k\|^2 := \int_{-\pi}^{\pi} |f(x) - P_k(x)|^2 dx$$

$S_{f,k}(x)$ **best approximation** (least error):

$$\|f - S_{f,k}\|^2 \leq \|f - P_k\|^2 \quad \text{for all } P_k(x)$$

$$\text{Obs: } \|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^k (a_n^2 + b_n^2) \right]$$

- 3 **Parseval's identity**

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx \quad (\text{when } \int_{-\pi}^{\pi} f^2 dx < \infty)$$

Summary: Fourier series

- 5 **Fourier series** of $2L$ -periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

- 6 **Complex Fourier series** of $2L$ -periodic $f(x)$:

$$S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}}$$

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-i \frac{n\pi x}{L}} dx$$

OBS: $\sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}} = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) !!!$