Summary: Fourier series

1. Fourier series of 2π-periodic f:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$S_{f,k}(x) = a_0 + \sum_{n=1}^{k}(a_n \cos nx + b_n \sin nx) \qquad k\text{-th partial sum}$$

2. Approximation of f by trigonometric polynomial

$$P_k(x) = A_0 + \sum_{n=1}^{k}(A_n \cos nx + B_n \sin nx)$$

Mean square (or L^2) error:

$$\boxed{\|f - P_k\|^2 := \int_{-\pi}^{\pi} |f(x) - P_k(x)|^2 dx}$$

$S_{f,k}(x)$ best approximation (least error):

$$\boxed{\|f - S_{f,k}\|^2 \leq \|f - P_k\|^2} \quad \text{for all } P_k(x)$$

Obs: $\|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{k}(a_n^2 + b_n^2)\right]$

3. Parseval's identity

$$\boxed{2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx} \quad \left(\text{when } \int_{-\pi}^{\pi} f^2 dx < \infty\right)$$

Summary: Fourier series

5. Real Fourier series of $2L$-periodic f:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

6. Complex Fourier series of $2L$-periodic f:

$$\boxed{S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}}}, \quad \boxed{c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{n\pi x}{L}} dx}$$

OBS: $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}} = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$!!!

Lecture 8: Fourier integrals and transforms

Kreyszig: Section 11.7, 11.9

1. Fourier integral
2. Fourier transform
3. Properties
4. Examples

Homework: Repeat complex *numbers*, *absolute values*, *exponentials* [Mat 3/Lin. Alg.]

Fourier integral of $f(x)$, $x \in \mathbb{R}$:

$$I_f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw,$$

where

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Fourier transform of $f(x)$, $x \in \mathbb{R}$:

$$\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Invers:

$$\mathcal{F}^{-1}[g](x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{iwx} dw$$

Properties:

$$\mathcal{F}[af(x) + bg(x)](w) = a\mathcal{F}[f](w) + b\mathcal{F}[f](w)$$

$$\mathcal{F}[f'](w) = (iw)\mathcal{F}[f](w) \qquad (|f(x)| \underset{|x|\to\infty}{\longrightarrow} 0)$$

$$\mathcal{F}[e^{-iax}f(x)](w) = \mathcal{F}[f](w+a)$$

Summary: Fourier integral and transform

1. Fourier integral of $f(x)$, $x \in \mathbb{R}$:

$$\boxed{I_f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\, dw} \quad \boxed{\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx}$$

2. Fourier transform of $f(x)$, $x \in \mathbb{R}$:

$$\boxed{\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx}$$

Invers: $$\boxed{\mathcal{F}^{-1}[g](x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{iwx}\, dw}$$

Under certain conditions: $f(x) = I_f(x) = \mathcal{F}^{-1}[\mathcal{F}[f]](x)$

3. Properties:

$$\mathcal{F}[af(x) + bg(x)](w) = a\mathcal{F}[f](w) + b\mathcal{F}[f](w)$$

$$\mathcal{F}[f'](w) = (iw)\mathcal{F}[f](w) \qquad (|f(x)| \to 0 \text{ as } |x| \to \infty)$$

$$\mathcal{F}[e^{-iax} f(x)](w) = \mathcal{F}[f](w + a)$$