Summary: Separation of variables – wave equation

(1)	$u_{tt} = c^2 u_{xx}$	$t > 0,\ x \in (0, L)$
(2)	$u(0, t) = 0 = u(L, t)$	$t > 0,\ x = 0, L$
(3)	$u(x, 0) = f(x),\ u_t(x, 0) = g(x)$	$t = 0,\ x \in (0, L)$

1. Find all solutions $u(x, t) = F(x)G(t)$ of (1) and hom. cond'ns (2):

$$\underset{(1),(2)}{\Rightarrow} \boxed{F'' - kF = 0} \boxed{G'' - c^2 kG = 0} \boxed{F(0) = 0 = F(L)}$$

$$\underset{u \neq 0}{\Rightarrow} \boxed{k = -(\tfrac{n\pi}{L})^2} \boxed{F_n(x) = \sin \tfrac{n\pi x}{L}} \boxed{G_n(t) = B_n \cos \tfrac{cn\pi t}{L} + B_n^* \sin \tfrac{cn\pi t}{L}}$$

2. Superposition and inhomogeneous conditions (3):

$$\boxed{u(x, t) = \sum_{n=0}^{\infty} F_n G_n = \sum_{n=0}^{\infty} (B_n \cos \tfrac{cn\pi t}{L} + B_n^* \sin \tfrac{cn\pi t}{L}) \sin \tfrac{n\pi x}{L}}$$

$$f(x) \overset{(3)}{=} u(x, 0) = \sum_{n=0}^{\infty} B_n \sin \tfrac{n\pi x}{L} \underset{F-\sin\ series}{\Rightarrow} \boxed{B_n = \tfrac{2}{L} \int_0^L f \sin \tfrac{n\pi x}{L} dx}$$

$$g(x) \overset{(3)}{=} u_t(x, 0) = \sum_{n=0}^{\infty} B_n^* \tfrac{cn\pi}{L} \sin \tfrac{n\pi x}{L} \Rightarrow \boxed{B_n^* \tfrac{cn\pi}{L} = \tfrac{2}{L} \int_0^L g \sin \tfrac{n\pi x}{L} dx}$$

Lecture 11: Partial differential equations

Kreyszig: Section 12.6

1. PDEs: Heat equation, Laplace equation
2. Boundary value problems: Cauchy, Dirichlet, Neumann.
3. Solution technique: Separation of variables

Homework:

1. Repeat *Solution of ordinary differential equations* [Mat 3]

Lecture 11: Separation of var's – heat equation

$$
\begin{array}{lll}
(1) & u_t = c^2 u_{xx} & t > 0,\ x \in (0, L) \\
(2) & u(0,t) = 0 = u(L,t) & t > 0,\ x = 0, L \\
(3) & u(x,0) = f(x) & t = 0,\ x \in (0, L)
\end{array}
$$

1. Find all solutions $u(x,t) = F(x)G(t)$ of (1) and (2):

$$\underset{(1),(2)}{\Rightarrow} \quad \boxed{F'' - kF = 0} \quad \boxed{G' - c^2 kG = 0} \quad \boxed{F(0) = 0 = F(L)}$$

$$\underset{u \neq 0}{\Rightarrow} \quad \boxed{k = -(\tfrac{n\pi}{L})^2} \quad \boxed{F_n(x) = \sin \tfrac{n\pi x}{L}} \quad \boxed{G_n(t) = B_n e^{-(\frac{cn\pi}{L})^2 t}}$$

2. Superposition and inhomogeneous condition (3):

$\boxed{u(x,t) = \sum_{n=0}^{\infty} F_n G_n}$ ... use (3) and Fourier sin series ...

Lecture 11: Separation of var's – Laplace equation

$$
\begin{array}{lll}
(1) & u_{xx} + u_{yy} = 0 & x \in (0,a),\ y \in (0,b) \\
(2) & u(0,y) = 0 = u(a,y) & y \in [0,b] \\
(3) & u(x,0) = 0, u(x,b) = f(x) & x \in [0,a]
\end{array}
$$

1. Find all solutions $u(x,y) = F(x)G(y)$ of (1), (2), and $(3)_1$:

$$\Rightarrow F'' - kF = 0, G'' + kG = 0, F(0) = 0 = F(a), G(0) = 0$$

$$\Rightarrow \boxed{k = -(\tfrac{n\pi}{a})^2} \ \boxed{F_n(x) = \sin \tfrac{n\pi x}{a}} \ \boxed{G_n(y) = 2A_n \sinh \tfrac{n\pi y}{a}}$$

2. Superposition and inhomogeneous condition $(3)_2$:

$\boxed{u(x,y) = \sum_{n=0}^{\infty} F_n G_n}$ ... use $(3)_2$ and Fourier sin series ...

Summary: Heat and Laplace equation

1. Boundary value problems:

   Cauchy u given at $t = 0$.
   Dirichlet u given on boundary
   Neumann (normal) derivative of u given on boundary

2. Heat equation:

(1)	$u_t = c^2 u_{xx}$	$t > 0, \ x \in (0, L)$
(2)	$u(0, t) = 0 = u(L, t)$	$t > 0, \ x = 0, L$
(2')	$u_x(0, t) = 0 = u_x(L, t)$	$t > 0, \ x = 0, L$
(3)	$u(x, 0) = f(x)$	$t = 0, \ x \in (0, L)$

   u temperature of rod, ends: fixed temperature (2) or insulated $(2')$

   Cauchy-Dirichlet $(1), (2), (3)$; Cauchy-Neumann $(1), (2'), (3)$

3. Laplace equation:

   (4) $$u_{xx} + u_{yy} = 0$$

   Electrostatic potential, potiential flow, membrane, temperature ...

Summary: Separation of variables – heat equation

(1)	$u_t = c^2 u_{xx}$	$t > 0,\ x \in (0, L)$
(2')	$u_x(0,t) = 0 = u_x(L,t)$	$t > 0,\ x = 0, L$
(3)	$u(x,0) = f(x)$	$t = 0,\ x \in (0, L)$

1. Find all solutions $u(x,t) = F(x)G(t)$ of (1) and hom. cond'ns $(2')$:

$$\underset{(1),(2')}{\Rightarrow} \boxed{F'' - kF = 0} \boxed{G' - c^2 kG = 0} \boxed{F'(0) = 0 = F'(L)}$$

$$\underset{u \neq 0}{\Rightarrow} \boxed{k = -(\tfrac{n\pi}{L})^2} \boxed{F_n(x) = \cos \tfrac{n\pi x}{L}} \boxed{G_n(t) = B_n e^{-(\frac{cn\pi}{L})^2 t}}$$

2. Superposition and inhomogeneous condition (3):

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} F_n G_n = \sum_{n=0}^{\infty} B_n e^{-(\frac{cn\pi}{L})^2 t} \cos \tfrac{n\pi x}{L}}$$

$$f(x) \overset{(3)}{=} u(x,0) = \sum_{n=0}^{\infty} B_n \cos \tfrac{n\pi x}{L}$$

$$\underset{F-\cos\ \text{series}}{\Rightarrow} \boxed{B_0 = \tfrac{1}{L} \int_0^L f\, dx} \boxed{B_n = \tfrac{2}{L} \int_0^L f \cos \tfrac{n\pi x}{L} dx}$$