Summary: Heat and Laplace equation

1. Boundary value problems:

   Cauchy u given at $t = 0$.
   Dirichlet u given on boundary
   Neumann (normal) derivative of u given on boundary

2. Heat equation:

(1)	$u_t = c^2 u_{xx}$	$t > 0,\ x \in (0, L)$
(2)	$u(0,t) = 0 = u(L,t)$	$t > 0,\ x = 0, L$
(2')	$u_x(0,t) = 0 = u_x(L,t)$	$t > 0,\ x = 0, L$
(3)	$u(x,0) = f(x)$	$t = 0,\ x \in (0, L)$

   u temperature of rod, ends: fixed temperature (2) or insulated $(2')$

   Cauchy-Dirichlet $(1), (2), (3)$; Cauchy-Neumann $(1), (2'), (3)$

3. Laplace equation: $u_{xx} + u_{yy} = 0$

   Electrostatic potiential, potiential flow, membrane, temperature ...

4. Solved by separation of variables, $u = F(x)G(t)$ ...

Review: Fourier transform

1. $$\boxed{\mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx}$$

2. $\mathcal{F}[f'](w) = (iw)\mathcal{F}[f](w)$

3. $\mathcal{F}[e^{-iax} f(x)](w) = \mathcal{F}[f](w+a)$

4. $\mathcal{F}^{-1}[e^{iaw} \hat{f}(w)](x) = f(x+a)$

5. $\mathcal{F}[f * g](w) = \sqrt{2\pi}\, \mathcal{F}[f](w) \cdot \mathcal{F}[g](w)$

   where $(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y) dy$

6. $\mathcal{F}[e^{-ax^2}](w) = \dfrac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$

Lecture 12: Partial differential equations

Kreyszig: Section 12.4, 12.5, 12.7

1. PDEs:

   Heat equation (with derivation)

   Wave equation

2. Solution techniques:

   Fourier transform

   Method of characteristics (D'Alembert)

Lecture 12: The heat equation in IR

$$(1) \qquad \begin{cases} u_t = c^2 u_{xx} & t > 0,\ x \in \mathbb{R} \\ u(x,0) = f(x) & t = 0,\ x \in \mathbb{R} \end{cases}$$

1. Derivation

2. Solution using Fourier transform

   (a) Fourier transform problem

   $$\hat{u}_t = -c^2 w^2 \hat{u}, \quad \hat{u}(x,0) = \hat{f}(x)$$

   (b) Solve

   $$\hat{u}(w,t) = \hat{f}(w) e^{-c^2 w^2 t}$$

   (c) Inverse Fourier transform

   $$u(x,t) = (f * g)(x), \ \ g = \mathcal{F}^{-1}[e^{-c^2 w^2 t}]$$

Lecture 12: The wave equation in IR

$$
\begin{aligned}
&(3) \qquad u_{tt} = c^2 u_{xx} && t > 0,\ x \in \mathbb{R} \\
&(6) \qquad u(x,0) = f(x) && t = 0,\ x \in \mathbb{R} \\
&(7) \qquad u_t(x,0) = g(x) && t = 0,\ x \in \mathbb{R}
\end{aligned}
$$

1. Solution by Fourier transform
2. D'Alembert's solution

$$y = x + ct, \quad s = x - ct, \quad v(y,s) = u(x,t)$$

$$\Longrightarrow v_{ys} = 0 \underset{integrate}{\Longrightarrow} v(y,s) = \phi(y) + \psi(s)$$

$$\Longrightarrow \boxed{u(x,t) = \phi(x+ct) + \psi(x-ct)}$$

Partial differential equations

1. Heat equation: $u_t = c^2 u_{xx}$

   Derived from Fourier's law and conservation of energy

2. Solution by Fourier transform:

$$
\begin{aligned}
& u_t = c^2 u_{xx}, \qquad u(x,0) = f(x) \\
\overset{\mathcal{F}}{\Longrightarrow}\quad & \hat{u}_t = -c^2 w^2 \hat{u}, \quad \hat{u}(w,0) = \hat{f}(w) \\
\overset{solve}{\Longrightarrow}\quad & \hat{u}(w,t) = \hat{f}(w) e^{-c^2 w^2 t} \\
\overset{\mathcal{F}^{-1}}{\Longrightarrow}\quad & u(x,t) = (f * g)(x,t) = \int_{-\infty}^{\infty} f(y) g(x-y)\, dy,
\end{aligned}
$$

where $g(x) = \mathcal{F}^{-1}[e^{-c^2 w^2 t}](x) = \frac{1}{\sqrt{4\pi c^2 t}} e^{-\frac{x^2}{4c^2 t}}$

3. D'Alembert's solution of wave equation

$$
u_{tt} = c^2 u_{xx} \underset{\substack{y = x + ct, \\ s = x - ct, \\ v(y,s) = u(x,t)}}{\Longrightarrow} v_{ys} = 0 \underset{integrate}{\Longrightarrow} v(y,s) = \phi(y) + \psi(s)
$$

$$
\boxed{u(x,t) = v(x+ct, x-ct) = \phi(x+ct) + \psi(x-ct)}
$$