Summary Lecture 18: Complex line integrals

1. Complex line integral:

   $\int_C [af(z) + bg(z)]dz = a\int_C f(z)dz + b\int_C g(z)dz$

   $\int_{-C} f(z)dz = -\int_C f(z)dz$

   $\int_{C_1 \cup C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$ when $C_1 \cap C_2 = \emptyset$

   $\boxed{|\int_C f(z)dz| \le M \cdot L}$ $M = \max_{z \in C} |f(z)|$, $L =$ length of C

2. Cauchy's integral theorem

   f analytic in simply connected domain D, $C \subset D$ simple, closed curve

   $$\Rightarrow \oint_C f(z)dz = 0$$

3. Consequences:

   (a) $\int_C f(z)dz$ is independent of path in D

   (b) The indefinite integral of f exists in D, i.e. a function F s.t.

   $$F'(z) = f(z) \qquad \text{and} \qquad \int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1).$$

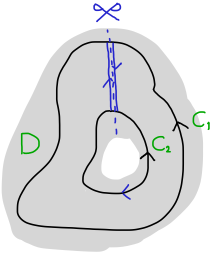
Summary Lecture 18: Domains with holes

4. Domains with holes:

   Cut to have a simlply connected domain...

   ... add segments along cut to have closed curve...

   Then use Cauchy:



Cauchy in the cut domain D^* and cancelations along cut:

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

Summary Lecture 18: Example

C_1 any simple, closed curve surrounding $z = z_0$

C_1 : circle $|z - z_0|^2 = r^2$

Cauchy's integral theorem in domain with one hole, $m \in \mathbb{Z}$:

$$\oint_{C_1} (z - z_0)^m dz = \oint_{C_2 : z(t) = z_0 + re^{it}} (z - z_0)^m dz$$

$$\underset{Eks.1}{\overset{\text{last time}}{=}} \begin{cases} 2\pi i, & m = -1, \\ 0, & m \neq -1, \; m \in \mathbb{Z} \end{cases}$$

Lecture 19: Complex Analysis

Kreyszig: Sections 14.3, 14.4

1. Cauchy integral formula
2. Analytic functions infinitely differentiable
3. Properties of analytic functions:
   Cauchys inequality and Liouvilles theorem
4. Examples

Homework:

Repeat from Mat 1/GKA 2: Sequences, series, convergence tests

Lecture 19: Cauchy's integral formula

(A1) f is analytic in simply connected domain D

(A2) $z_0 \in D$, $C \subset D$ simple closed curve, positively oriented, enclosing z_0

Then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

Lecture 19: Analytic => infinitly differentiable

f analytic in D $\Rightarrow$ f ∞-differentiable in D, and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Lecture 19: Properties of analytic functions

Cauchy's inequality:

$$f \text{ analytic} \quad \Longrightarrow \quad |f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|$$

Gauss mean value theorem:

$$f \text{ analytic} \quad \Longrightarrow \quad f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

Liouville's theorem:

$$f \text{ analytic, bounded in } \mathbb{C} \quad \Rightarrow \quad f \text{ is constant}$$

Lecture 19: Properties of analytic functions

Cauchy's inequality:

$$f \text{ analytic in } |z-z_0| \le r \quad \Longrightarrow \quad |f^{(n)}(z_0)| \le \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|$$

Gauss mean value theorem:

$$f \text{ analytic in } |z-z_0| \le r \quad \Longrightarrow \quad f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

Liouville's theorem:

f analytic, bounded in $\mathbb{C}$ $\Longrightarrow$ f is constant

Maximum (modulus) principle:

f analytic in domain D and $|f(z)|$ attains its max in D

$\Longrightarrow$ f is constant in D

Morera's theorem:

f continuous in D and $\oint_C f(z)dz = 0$ for all simple, closed $C \subset D$

$\Longrightarrow$ f analytic in D

Summary Lecture 19

1. Cauchy's integral formula: $$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz}$$ if

   (A1) f is analytic in simply connected domain D

   (A2) $z_0 \in D$, $C \subset D$ simple closed curve, positively oriented, enclosing z_0.

2. Infinitly differentiable:

   f analytic in D $\Rightarrow$ f infinitely differentiable in D, and

   $$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz}$$

3. Properties of analytic functions:

   Cauchy's inequality: $$\boxed{|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{|z - z_0| = r} |f(z)|}$$ if f analytic

   Liouville's theorem: f analytic, bounded in $\mathbb{C}$ $\Rightarrow$ f is constant

   Morera's theorem: f continuous in D and $\oint_C f(z) dz = 0$ for all simple, closed $C \subset D$ $\Rightarrow$ f analytic in D