Summary Lecture 21: Complex power series

1. Termwise operations on power series:

(a) $$c_1 \sum_{n=0}^{\infty} a_n z^n + c_2 \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n) z^n \qquad [R = \min(R_a, R_b)]$$

(b) $$\left(\sum_{n=0}^{\infty} a_n z^n \right)' = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad [R = R_a]$$

(c) $$\int \left(\sum_{n=0}^{\infty} a_n z^n \right) dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} + C \qquad [R = R_a]$$

2. Representation of functions by power series:

Represents their sum where they converges

Uniqueness: $$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n \quad \Rightarrow \quad a_n = b_n, \ n = 0, 1, 2, \dots$$

The function is analytic where the power series converges

Summary Lecture 21: Complex power series

3. Taylor series:

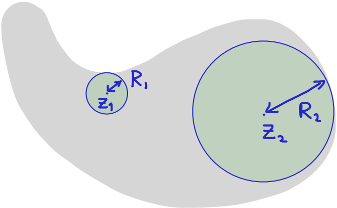
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

4. Taylor's theorem:

If f is analytic in a domain D and $z_0 \in D$, then

(a) the Taylor series of f about z_0 exists and is unique,

(b) and it converges and equals f in the largest open disc about z_0 where f is analytic



Lecture 22: Taylor and Laurent series

Kreyszig: Sections 15.4, 16.1

1. Important power series
2. Finding power series
3. Laurent series, definition and convergence
4. Examples

Homework:

Repeat *Taylor series* [Mat 1/GKA 1]

Important series:

e^z, $\cos z$, $\sin z$, $\cosh z$, ... , $\mathrm{Ln} z$, ..., $\frac{1}{1-z}$, ...

Finding Taylor series: (as for real series)

Use known series, term wise operations (addition, derivation, integration), substitution, geometric series, binomial series, partial fractions.

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Example:

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots$$

(a) There exists a unique Laurent series (1) that converges to f in the largest annulus

$$D : r < |z - z_0| < R$$

where f is analytic.

(b) $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*, \quad b_n = a_{-n}.$

Summary Lecture 22: Taylor and Laurent series

1. Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (1)$$

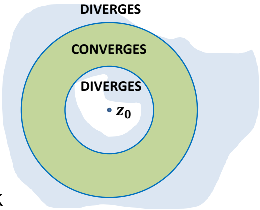
Example: $e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots$

2. Laurent's theorem:

(a) There exists a unique Laurent series (1) that converges to f in the largest annulus $D : r < |z - z_0| < R$ where f is analytic.

(b) $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$,

$b_n = \frac{1}{2\pi i} \oint_C f(z^*)(z^* - z_0)^{n-1} dz^*$



3. Remarks:

Term wise addition, differerentiation, . . . is ok

a_n, b_n found from known Taylor series, substitution, . . .