Summary Lecture 22: Taylor and Laurent Series

1. Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (1)$$

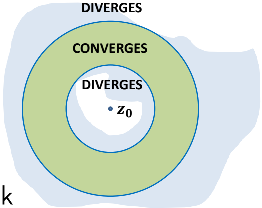
Example: $e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots$

2. Laurent's theorem:

(a) There exists a unique Laurent series (1) that converges to f in the largest annulus $D : r < |z - z_0| < R$ where f is analytic.

(b) $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$,

$b_n = a_{-n}$.



3. Remarks:

Term wise addition, differerentiation, ... is ok

a_n, b_n found from known Taylor series, substitution, ...

Lecture 23: Singularities and Residue integration

Kreyszig: Sections 16.2, 16.3

1. Singular points
2. Zeros
3. Classification, properties, $z = \infty$
4. Residue integration
5. Examples

Points z_0 where $f(z)$ is not analytic/defined

Isolated singularity z_0, the only one in some nbhd

$\Downarrow$ **Laurent's theorem**

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\neq 0}, \quad 0 < |z - z_0| < R$$

Principal value of z_0: $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

z_0 order n pole: $b_n \neq 0,\ b_k = 0,\ k > n$

z_0 isolated essential singularity: $b_n \neq 0$ for ∞ many n

Zeros of order m

Zeros are isolated; f has zero $\implies$ $1/f$ has pole

$f(z)$ has pole/essential singularity/zero at $z = \infty$

$\Updownarrow$ DEF

$f(\frac{1}{w})$ has pole/essential singularity/zero at $w = 0$

z_0 only singularity of $f(z)$ enclosed by C

$\Downarrow$ **Laurent's theorem**

$$f(z) = a_0 + a_1(z - z_0) + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots,$$

$$(0 < |z - z_0| < R)$$

$\Downarrow$ **Term wise integration/Laurent's theorem**

$$\boxed{\oint_C f(z)dz = 2\pi i b_1}$$

Residue of $f(z)$ at z_0: $\quad \operatorname*{Res}_{z=z_0} f(z) = b_1$

Summary Lecture 23: Singularities

1. Singularity: Point z_0 where $f(z)$ is not analytic/defined $(\dots)$

2. Isolated singularity:

   z_0 is the only singularity in $|z - z_0| < R$ for some $R > 0$

   $\Downarrow$ **Laurent's theorem**

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\neq 0}, \quad 0 < |z - z_0| < R$$

Principal value of z_0: $\boxed{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}$

z_0 order n pole: $b_n \neq 0,\ b_k = 0,\ k > n$

z_0 isolated essential singularity: $b_n \neq 0$ for ∞ many n

Summary Lecture 23: Residue integration

3. Residue integration:

   Goal: $\oint_C f(z)dz = 2\pi i \sum \text{residues}$

4. Only one singularity:

   z_0 only singularity of $f(z)$ enclosed by C (simple, closed, counter cl.wise)

   $\Downarrow$ **Laurent's theorem**

   $$f(z) = a_0 + a_1(z - z_0) + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots, \; 0 < |z - z_0| < R$$

   $\Downarrow$ **Term wise integration/Laurent's theorem**

   $$\oint_C f(z)dz = 2\pi i b_1$$

5. Residue of $f(z)$ at z_0: $\operatorname*{Res}_{z=z_0} f(z) = b_1$

   b_1-coefficient in Laurent series that converges in $0 < |z - z_0| < R$!