# Summary Lecture 24: Residue integration

1. Residues: $\operatorname*{Res}_{z=z_0} f(z) = b_1$ where

$$f(z) = \sum a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots,\ 0 < |z-z_0| < R$$

$z_0$ order 1 pole:

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z) \quad \text{or} \quad \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

$z_0$ order $n$ pole:

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!}\lim_{z\to z_0}\Big((z-z_0)^n f(z)\Big)^{(n-1)}$$

$z_0$ isolated essential singularity:

Find $b_1$ from the Laurent series!

2. Residue theorem: Assume

(A1) $C$ simple, closed curve oriented counterclockwisely

(A2) $f(z)$ has finite number of singularities $z_1, \ldots, z_m$ enclosed by $C$

Then
$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} f(z)$$

# Lecture 25: Residue integration of real integrals

Kreyszig: Section 16.4

1. Computing real integrals:

   Type I: $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

   Type II: $\int_{-\infty}^{\infty} f(x)\, dx$

   Type III: $\int_{-\infty}^{\infty} f(x) e^{-iwz}\, dx$

2. Examples

**Information (check www):**

Øving 13, 8/13 øvinger for å ta eksamen, neste uke

**Frist øving 12 og 13:**

Denne uka! Onsdag kl 23:59!

# Lecture 25: Type I real integrals

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta, \quad F \text{ rational}$$

Substitution:

$$z = e^{i\theta}$$

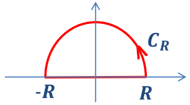
$$\cos\theta = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right), \qquad \sin\theta = \tfrac{1}{2i}\left(z - \tfrac{1}{z}\right)$$

$$dz = iz\, d\theta$$

$$\text{Interval } [0, 2\pi) \ \rightsquigarrow \ \text{circle } |z| = 1$$

$I = \int_{-\infty}^{\infty} f(x)\,dx,\quad f$ rational, no real singularities

(i) $I = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$




(ii) $\int_{-R}^{R} f(x)\,dx = \oint_{C_R} f(z)\,dz - \int_{S_R} f(z)\,dz$

(iii) $R$ big enough $\Rightarrow$ $C_R$ encircles all poles in upper half plane

$$\oint_{C_R} f(z)dz \overset{\text{Residue thm.}}{=} 2\pi i \sum_{\substack{\text{poles in upper}\\ \text{half plane}}} \operatorname{Res} f(z)$$

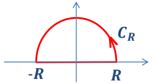
(iv) $|\int_{S_R} f(z)dz| \le \max_{z\in S_R} |f(z)|\cdot L \le \max_{|z|=R} |f(z)|\cdot \pi R \xrightarrow[R\to\infty]{\text{Must show}} 0$

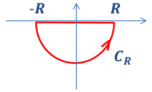
# Lecture 25: Type III real integrals

$$I = \int_{-\infty}^{\infty} f(x) e^{-iwx} dx, \quad f \text{ rational, no real sing.}$$

Same proceedure as for Type II integrals...

...and choose $C_R$ to insure $w \cdot \operatorname{Im} z \leq 0$:

$w \leq 0 \rightsquigarrow$ 

; $w > 0 \rightsquigarrow$ 


$$\Longrightarrow |f(z)e^{-iwz}| \overset{z=x+iy}{=} |f(z)|e^{w \cdot y} \overset{z \in C_R}{\leq} |f(z)|$$

$$\boxed{\operatorname{Re} I = \int_{-\infty}^{\infty} f(x) \cos(wx)\, dx} \quad \boxed{\operatorname{Im} I = -\int_{-\infty}^{\infty} f(x) \sin(wx)\, dx}$$

# Summary Lecture 25: Real integrals

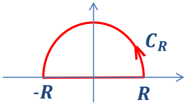
1. Type I: $\boxed{I = \int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta}$ $F$ rational

   Substitution: $z = e^{i\theta}$, $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$,..., $[0, 2\pi) \rightsquigarrow |z| = 1$

2. Type II: $\boxed{I = \int_{-\infty}^{\infty} f(x)\, dx}$ $f$ rational, no real singularities, ...

   (i) $I = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx$

   (ii) $\int_{-R}^{R} f(x)\, dx = \oint_{C_R} f(z)\, dz - \int_{S_R} f(z)\, dz$




   (iii) $R$ big enough $\Rightarrow$ $C_R$ encircles all poles in upper half plane

$$\oint_{C_R} f(z)\, dz \overset{\text{Residue thm.}}{=} 2\pi i \sum_{\substack{\text{poles in upper} \\ \text{half plane}}} \mathrm{Res}\, f(z)$$

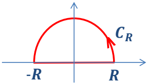
   (iv) $|\int_{S_R} f(z)\, dz| \le \max_{z\in S_R} |f(z)| \cdot L \le \max_{|z|=R} |f(z)| \cdot \pi R \xrightarrow[R\to\infty]{\text{Must show}} 0$

3. Type III: $\boxed{I = \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx}$ $f$ rational, no real singularities, ...

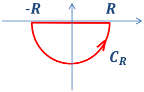
Same proceedure as for Type II integrals...

...and choose $C_R$ to insure $\boxed{w \cdot \operatorname{Im} z \leq 0}$:

$w \leq 0 \rightsquigarrow$




; $w > 0 \rightsquigarrow$




$$\implies |f(z)e^{-iwz}| \overset{z=x+iy}{=} |f(z)|e^{w \cdot y} \overset{z \in C_R}{\leq} |f(z)|$$

4. $\boxed{\operatorname{Re} I = \int_{-\infty}^{\infty} f(x) \cos(wx)\, dx}$ $\boxed{\operatorname{Im} I = -\int_{-\infty}^{\infty} f(x) \sin(wx)\, dx}$