Summary Lecture 25: Real integrals

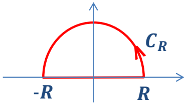
1. Type I: $I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$ F rational

   Substitution: $z = e^{i\theta}$, $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$,..., $[0, 2\pi) \rightsquigarrow |z| = 1$

2. Type II: $I = \int_{-\infty}^{\infty} f(x)\,dx$ f rational, no real singularities, ...

   (i) $I = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$

   (ii) $\int_{-R}^{R} f(x)\,dx = \oint_{C_R} f(z)\,dz - \int_{S_R} f(z)\,dz$



   (iii) R big enough $\Rightarrow$ C_R encircles all poles in upper half plane

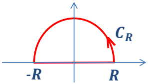
$$\oint_{C_R} f(z)\,dz \overset{\text{Residue thm.}}{=} 2\pi i \sum_{\substack{\text{poles in upper}\\ \text{half plane}}} \mathrm{Res}\, f(z)$$

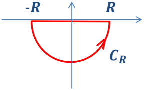
   (iv) $\left|\int_{S_R} f(z)\,dz\right| \le \max_{z\in S_R} |f(z)| \cdot L \le \max_{|z|=R} |f(z)| \cdot \pi R \xrightarrow[R\to\infty]{\text{Must show}} 0$

3. Type III: $\boxed{I = \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx}$ f rational, no real singularities, ...

Same proceedure as for Type II integrals...

...and choose C_R to insure $\boxed{w \cdot \mathrm{Im}\, z \leq 0}$:

$w \leq 0 \rightsquigarrow$ 

; $w > 0 \rightsquigarrow$ 

$$\Longrightarrow |f(z) e^{-iwz}| \overset{z=x+iy}{=} |f(z)| e^{w \cdot y} \overset{z \in C_R}{\leq} |f(z)|$$

4. $\boxed{\mathrm{Re}\, I = \int_{-\infty}^{\infty} f(x) \cos(wx)\, dx}$ $\boxed{\mathrm{Im}\, I = -\int_{-\infty}^{\infty} f(x) \sin(wx)\, dx}$

Lecture 26: Real integrals and singularities

Kreyszig: Section 16.4

1. Computing real integrals:

   Type IV: $\int_{-\infty}^{\infty} f(x)dx$, f has order 1 poles on the real axis

2. Examples

Frist øving 12 og 13:

Denne uka! Onsdag kl 23:59!

Next week:

Exam problems, repetition

Lecture 26: Singular integrals, principal values

$f(x)$ singular at $x_0 \in (a, b)$:

$$\int_a^b f(x)dx := \left(\lim_{r\to 0^+} \int_a^{x_0-r} + \lim_{r\to 0^+} \int_{x_0+r}^b \right) f(x)\, dx$$

$$\text{pr.v.} \int_a^b f(x)dx := \lim_{r\to 0} \left(\int_a^{x_0-r} + \int_{x_0+r}^b \right) f(x)\, dx$$

Integral exists $\underset{\not\Leftarrow}{\Rightarrow}$ principal value exisits

Lecture 26: Type IV real integrals

$$I = \mathrm{pr.v.} \int_{-\infty}^{\infty} f(x)\,dx, \; f \text{ rational, order 1 poles on } \mathbb{R}$$

(i) $$I = \lim_{\substack{R \to \infty \\ r \to 0}} \left(\int_{-R}^{a_1 - r} + \sum_{j=1}^{m-1} \int_{a_j + r}^{a_{j+1} - r} + \int_{a_m + r}^{R} \right) f(x)\,dx$$

(ii) $$\int_{-R}^{R} f(x)\,dx = \oint_{C_{r,R}} f(z)\,dz - \left(\int_{S_R} + \sum_{j=1}^{m} \int_{S_{j,r}} \right) f(z)\,dz$$

(iii) $$\oint_{C_{r,R}} f(z)\,dz \overset{\text{Residue thm.}}{=} 2\pi i \sum_{\substack{\text{poles in upper} \\ \text{half plane}}} \mathrm{Res}\, f(z) \qquad (R \text{ big, } r \text{ small})$$

(iv) $$\left| \int_{S_R} f(z)\,dz \right| \leq \max_{|z|=R} |f(z)| \cdot \pi R \xrightarrow[R \to \infty]{\text{Must show}} 0$$

(v) Lemma: $$\int_{S_{j,r}} f(z)\,dz \xrightarrow[r \to 0]{} \pi i \operatorname*{Res}_{z = a_j} f(z)$$

Summary Lecture 26: Real integrals

1. Principal value:

   $f(x)$ singular at $x_0 \in (a,b)$

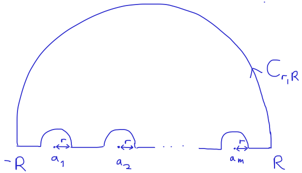
   $$\Longrightarrow \text{pr.v.} \int_a^b f(x)\,dx := \lim_{r\to 0}\left(\int_a^{x_0-r} + \int_{x_0+r}^b\right) f(x)\,dx$$

   Integral exists $\Rightarrow$ principal value exisits $\not\Rightarrow$ integral exists

2. Type IV: $\boxed{I = \int_{-\infty}^{\infty} f(x)\,dx}$ f rational, order 1 poles $a_1, \ldots, a_m \in \mathbb{R}$

   (i) $I = \lim_{\substack{R\to\infty \\ r\to 0}} \left(\int_{-R}^{a_1-r} + \sum_{j=1}^{m-1} \int_{a_j+r}^{a_{j+1}-r} + \int_{a_m+r}^{R}\right) f(x)\,dx$

   (ii) $\int_{-R}^{R} f(x)\,dx = \oint_{C_{r,R}} f(z)\,dz - \left(\int_{S_R} + \sum_{j=1}^{m} \int_{S_{j,r}}\right) f(z)\,dz$



Summary Lecture 26: Real integrals

(iii) R big and r small enough

$\Rightarrow C_{r,R}$ encircles all poles in upper half plane

$$\Rightarrow \oint_{C_{r,R}} f(z)dz \overset{\text{Residue thm.}}{=} 2\pi i \sum_{\substack{\text{poles in upper}\\ \text{half plane}}} \operatorname{Res} f(z)$$

(iv) $$\left|\int_{S_R} f(z)dz\right| \leq \max_{z\in S_R} |f(z)| \cdot L \leq \max_{|z|=R} |f(z)| \cdot \pi R \xrightarrow[R\to\infty]{\text{Must show}} 0$$

(v) Lemma: $$\int_{S_{j,r}} f(z)dz \underset{r\to 0}{\longrightarrow} \pi i \operatorname*{Res}_{z=a_j} f(z)$$