Summary: Laplace Transform

1. $F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}dt$

2. Unit step function: ($a \in \mathbb{R}$ fixed)

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases} \qquad \mathcal{L}[u(t-a)](s) = \frac{1}{s}e^{-as}$$

3. Properties:

Linearity: $\mathcal{L}[af(t) + bg(t)](s) = a\mathcal{L}[f](s) + b\mathcal{L}[g](s)$

s-shift: $\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a)$

t-shift: $\boxed{\mathcal{L}[f(t-a)u(t-a)](s) = e^{-as}\mathcal{L}[f](s)}$

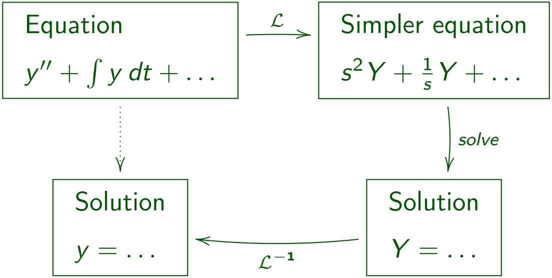
Derivatives: $\mathcal{L}[f'(t)](s) = s\mathcal{L}[f](s) - f(0)$

$\mathcal{L}[f''(t)](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0)$

Integral: $\boxed{\mathcal{L}[\int_0^t f(\tau)d\tau](s) = \frac{1}{s}\mathcal{L}[f](s)}$

Summary: Laplace Transform

5. Solving equations with the Laplace transform:



Lecture 3: Laplace Transform

Kreyszig: Sections 6.4 and 6.5

1. δ-functions
2. Convolutions
3. Integral representation of solutions of ODEs
4. On finding partial fractions
5. Many examples

Delta-function:

$$\text{“}\quad \delta(t-a) = \lim_{h\to 0} \frac{u(t-a) - u(t-(a+h))}{h} \quad\text{”}$$

Laplace-transform:

$$\mathcal{L}[\delta(t-a)](s) = e^{-as}$$

Convolution:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Laplace transform:

$$\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)$$

$$y'' + ay' + by = r(t), \quad t > 0,$$
$$y(0) = 0 = y'(0).$$

Integral representation:

$$y(t) = (q * r)(t) = \int_0^t q(t-\tau) r(\tau)\, d\tau$$

where

$$q(t) = \mathcal{L}^{-1}[\tfrac{1}{s^2+as+b}](t)$$

Partial fraction decomposition

$P(s)$ and $Q(s)$ polynomials, no common factor, $\mathrm{order}(P) < \mathrm{order}(Q)$

1. $Q(s) = (s - s_1)(s - s_2)(s - s_3)\dots$ non-repeated factors

   $$\Rightarrow \qquad \frac{P(s)}{Q(s)} = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \frac{A_3}{s - s_3} + \dots$$

2. $Q(s) = (s - s_0)^n \dots$ repeated factors

   $$\Rightarrow \qquad \frac{P(s)}{Q(s)} = \frac{A_1}{s - s_0} + \frac{A_2}{(s - s_0)^2} + \dots + \frac{A_n}{(s - s_o)^n} + \dots$$

3. $Q(s) = (s^2 + b_1 s + a_1)(s^2 + b_2 s + a_2)\dots$ irreducible, non-repeated quad. factors

   $$\Rightarrow \qquad \frac{P(s)}{Q(s)} = \frac{A_1 s + B_1}{s^2 + b_1 s + a_1} + \frac{A_2 s + B_2}{s^2 + b_2 s + a_2} + \dots$$

4. $Q(s) = (s^2 + b_1 s + a_1)^n \dots$ see earlier mathematics courses.

Partial fraction decomposition

Eksample 1:

$$\frac{s^5+2}{s(s+1)(s-1)^2(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{Es+F}{s^2+1}$$

Eksample 2:

$$\frac{1}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

Multiply by denominator $(s^2+1)(s^2+2s+2)$

$$\begin{aligned} 1 &= 1 + 0s + 0s^2 + 0s^3 \\ &= (As+B)(s^2+2s+2) + (Cs+D)(s^2+1) \\ &= (2B+D) + (2B+2A+C)s + (2A+B+D)s^2 + (A+C)s^3 \end{aligned}$$

Coefficients of same powers of s must coincide:

$$\begin{cases} 1 = 2B + D \\ 0 = 2B + 2A + C \\ 0 = 2A + B + D \\ 0 = A + C \end{cases} \Rightarrow \begin{cases} A = -\frac{2}{5} \\ B = \frac{1}{5} \\ C = \frac{2}{5} \\ D = \frac{3}{5} \end{cases}$$

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   Derivatives: $\mathcal{L}[f'(t)](s) = s\mathcal{L}[f](s) - f(0)$

   Integral: $\mathcal{L}[\int_0^t f(\tau)d\tau](s) = \frac{1}{s}\mathcal{L}[f](s)$

   Convolution: $\boxed{\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)}$ $\quad (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

3. Heaviside, delta functions:

   δ-function: $\int_0^\infty \delta(t-a)f(t)dt = f(a)$ for all continuous f.

   $\mathcal{L}[u(t-a)] = \frac{1}{s}e^{-as}$, $\boxed{\mathcal{L}[\delta(t-a)] = e^{-as}}$