Laplace Transform

1. $F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}dt$
2. Properties:

   Linearity: $\mathcal{L}[af(t) + bg(t)](s) = a\mathcal{L}[f](s) + b\mathcal{L}[g](s)$

   s-shift: $\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a)$

   t-shift: $\mathcal{L}^{-1}[e^{-as}F(s)](t) = f(t-a)u(t-a)$

   Derivatives: $\mathcal{L}[f'(t)](s) = s\mathcal{L}[f](s) - f(0)$

   Integral: $\mathcal{L}[\int_0^t f(\tau)d\tau](s) = \frac{1}{s}\mathcal{L}[f](s)$

   Convolution: $\boxed{\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)}$ $\quad (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$
3. Unit step function, delta function:

   δ-function: $\int_0^\infty \delta(t-a)f(t)dt = f(a)$ for all f cont. at $t = a\,(> 0)$

   $\mathcal{L}[u(t-a)] = \frac{1}{s}e^{-as}$, $\boxed{\mathcal{L}[\delta(t-a)] = e^{-as}}$

Lecture 4: Laplace Transform

Kreyszig: Sections 6.6, 6.7, 11.1

1. Differentiation of transforms
2. Integration of transforms
3. Systems of differential equations
4. Fourier series (introduction)
5. Examples

Derivatives and integrals of transforms:

$$-F'(s) = \mathcal{L}[tf(t)](s)$$

$$\int_s^\infty F(\bar{s})d\bar{s} = \mathcal{L}[\tfrac{1}{t}f(t)](s)$$

Systems of ordinary differential equations:

Equation

$$\vec{y}\,' + A\vec{y} = \vec{r}(t)$$

$\xrightarrow{\mathcal{L}}$

Simpler equation

$$s\vec{Y} - \vec{y}(0) + A\vec{Y} = \vec{R}$$

$\downarrow$ *solve*

Solution

$$\vec{Y} = \dots$$

$\xleftarrow{\mathcal{L}^{-1}}$

Solution

$$\vec{y} = \dots$$

$f(x)$ is p-periodic if $f(x+p) = f(x)$ for all $x \in \mathbb{R}$.

Representation of $f(x)$ by trigonometric series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big).$$

Summary: Laplace Transform

1. $F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}dt$
2. Properties:

   Linearity: $\mathcal{L}[af(t) + bg(t)](s) = a\mathcal{L}[f](s) + b\mathcal{L}[g](s)$

   s-shift: $\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a)$

   t-shift: $\mathcal{L}^{-1}[e^{-as}F(s)](t) = f(t-a)u(t-a)$

   Derivatives: $\mathcal{L}[f'(t)](s) = s\mathcal{L}[f](s) - f(0)$

   $$\boxed{\mathcal{L}^{-1}[F'(s)](t) = -tf(t)}$$

   Integral: $\mathcal{L}[\int_0^t f(\tau)d\tau](s) = \frac{1}{s}\mathcal{L}[f](s)$

   $$\boxed{\mathcal{L}^{-1}[\int_s^\infty F(\bar{s})d\bar{s}](t) = \frac{1}{t}f(t)}$$

   Convolution: $\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)$ $\quad (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

3. Unit step and delta function: $\mathcal{L}[u(t-a)] = \frac{1}{s}e^{-as}$, $\mathcal{L}[\delta(t-a)] = e^{-as}$