Fourier Series

1. $f(x)$ is p-periodic if $f(x+p) = f(x)$ for all $x \in \mathbb{R}$.

2. Fourier Series:

   Representation of periodic functions by trigonometric series,

   $$(1) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big) \qquad\qquad (p = 2\pi).$$

3. Questions:

   What is a_n and b_n?

   When does the series (1) converges?

   When and where is its sum equal $f(x)$?

4. Comments

   Fourier series can represent discontinuous functions!!

   Fourier series are a very important tool in science and technology.

Lecture 5: Fourier Series

Kreyszig: Section 11.1

1. Periodic functions
2. The trigonometric system
3. Fourier series, the coefficients
4. Fourier series, convergence and sum
5. Examples

p-periodic functions:

$$f(x) = f(x+p) \quad \text{for all} \quad x \in \mathbb{R}.$$

Trigonometric system:

$$\mathcal{T} = \{1, \cos x, \sin x, \dots, \cos nx, \sin nx, \dots\}$$

(2π-periodic)

Orthogonality:

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x) g(x) dx = 0$$

for $f, g \in \mathcal{T}$, $f \neq g$.

Fourier series for 2π-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx = \frac{\langle f, \cos nx \rangle}{\langle \cos nx, \cos nx \rangle}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx = \frac{\langle f, \sin nx \rangle}{\langle \sin nx, \sin nx \rangle}$$

Convergence and sum

Assume

(A1) f periodic and piecewise continuous

(A2) f has both right and left derivatives at x

Then the Fourier series S_f converge at x and

$$S_f(x) = \frac{f(x^+) + f(x^-)}{2},$$

i.e. $S_f(x) = f(x)$ if f is also continuous at x.

($\mathcal{T}$ orthogonal basis for L^2: $f \in L^2(-\pi,\pi) \implies f = \sum_{p\in\mathcal{T}} \frac{\langle f, p\rangle}{\langle p, p\rangle} p$)

Summary: Fourier series

1. Periodic functions:
   p-periodic: $f(x) = f(x+p) \;\; (= f(x+np),\; n \in \mathbb{N})\,, \quad x \in \mathbb{R}$.

   2π-periodic: $a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big)$

2. Trigonometric system:
   $\mathcal{T} = \{1, \cos x, \sin x, \ldots, \cos nx, \sin nx, \ldots\}$ (2π-periodic)

   Orthogonality:

   $$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x) g(x) dx = 0 \quad \text{for} \quad f, g \in \mathcal{T},\; f \neq g.$$

3. Fourier series for 2π-periodic $f(x)$:

   $S_f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big)$ where

   $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,$$

   $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx,$$

   $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx$$

Summary: Fourier series

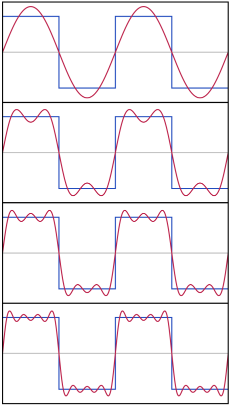

Figure: From Wikipedia

5. Convergence and sum

Assume

(A1) f periodic and piecewise continuous,

(A2) f has both right and left derivatives at x.

Then the Fourier series S_f converge at x and

$$S_f(x) = \frac{f(x^+) + f(x^-)}{2},$$

i.e. $S_f(x) = f(x)$ if f is also continuous at x.

`Animation by Yuya` [a]

[a] Oscillations near discontinuities do not die out – Gibb's phenomenon!