Summary: Fourier series

1. Fourier series of $p = 2L$-periodic $f(x)$:

$$\boxed{S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)}$$

where
$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx,$$
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx,$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

2. Odd and even functions

Even $g(-x) = g(x)$ $\quad \boxed{\int_{-L}^{L} g(x)dx = 2\int_0^L g(x)dx}$

Odd $h(-x) = -h(x)$ $\quad \boxed{\int_{-L}^{L} h(x)dx = 0}$

even · even = odd · odd = even; odd · even = odd

Summary: Fourier series

4. Fourier sin and cos series:

   f even: $S_f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$,

   $$\boxed{a_0 = \frac{1}{L}\int_0^L f(x)dx}, \quad \boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx}$$

   f odd: $S_f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$,

   $$\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx}$$

5. Even and odd $2L$-periodic extensions of $f(x)$, $0 \le x \le L$:

   $f_1(x)$, $x \in \mathbb{R}$, even, $2L$-periodic, $f_1 = f$ on $[0, L]$,

   $S_{f_1}(x) =$ cos-series $=:$ the Fourier cos series of f

   $f_2(x)$, $x \in \mathbb{R}$, odd, $2L$-periodic, $f_2 = f$ on $[0, L]$,

   $S_{f_2}(x) =$ sin-series $=:$ the Fourier sin series of f

6. **OBS:** $f = f_1 = f_2$ on $[0, L]$, $\quad f_1 \neq f_2$, $\quad f$ not defined on $[0, L]^c$!

Lecture 7: Fourier Series

Kreyszig: Section 11.4 (10th ed.) and Section 11.4 in 9th ed.!

1. Approximation with trigonometric polynomials
2. Bessel's inequality, Parseval's identity
3. Complex Fourier series
4. Examples

Section 11.4 in 9th ed. available on course wiki page (Fremdriftsplan).

Homework: Repeat complex *numbers*, *absolute values*, *exponentials* [Mat 3]

Approximation of $f(x)$ by trigonometric polynomial

$$P_k(x) = A_0 + \sum_{n=1}^{k} (A_n \cos nx + B_n \sin nx).$$

Mean square (L^2) error minimal when $P_k = S_{f,k}$:

$$\int_{-\pi}^{\pi} |f(x) - S_{f,k}(x)|^2 dx \leq \int_{-\pi}^{\pi} |f(x) - P_k(x)|^2 dx$$

for all $P_k(x)$

where the k-th Fourier partial sum $S_{f,k}$ is

$$S_{f,k}(x) = a_0 + \sum_{n=1}^{k} (a_n \cos nx + b_n \sin nx).$$

$$\|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{k}(a_n^2 + b_n^2)\right]$$

Bessel's inequality

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx$$

Parseval's identity

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx$$

Fourier series of $2L$-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

Complex Fourier series of $2L$-periodic $f(x)$:

$$S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{n\pi x}{L}} dx$$

Summary: Fourier series

1. Fourier series of 2π-periodic $f(x)$:

$$S_f(x) \;= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$S_{f,k}(x) = a_0 + \sum_{n=1}^{k}(a_n \cos nx + b_n \sin nx) \qquad k\text{-th partial sum}$$

2. Approximation of $f(x)$ by trigonometric polynomial

$$P_k(x) = A_0 + \sum_{n=1}^{k}(A_n \cos nx + B_n \sin nx)$$

Mean square (or L^2) error:

$$\boxed{\|f - P_k\|^2 := \int_{-\pi}^{\pi} |f(x) - P_k(x)|^2 dx}$$

$S_{f,k}(x)$ best approximation (least error):

$$\boxed{\|f - S_{f,k}\|^2 \leq \|f - P_k\|^2} \quad \text{for all } P_k(x)$$

Obs: $\|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{k}(a_n^2 + b_n^2)\right]$

3. Parseval's identity

$$\boxed{2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx} \quad \left(\text{when } \int_{-\pi}^{\pi} f^2 dx < \infty\right)$$

Summary: Fourier series

5. Fourier series of $2L$-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

6. Complex Fourier series of $2L$-periodic $f(x)$:

$$\boxed{S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}}} \qquad \boxed{c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{n\pi x}{L}} dx}$$

OBS: $\displaystyle \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}} = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$!!!