

Summary: Fourier series

- ① **Fourier series** of 2π -periodic f :

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$S_{f,k}(x) = a_0 + \sum_{n=1}^k (a_n \cos nx + b_n \sin nx) \quad k\text{-th partial sum}$$

- ② **Approximation of f by trigonometric polynomial**

$$P_k(x) = A_0 + \sum_{n=1}^k (A_n \cos nx + B_n \sin nx)$$

Mean square (or L^2) error:

$$\|f - P_k\|^2 := \int_{-\pi}^{\pi} |f(x) - P_k(x)|^2 dx$$

$S_{f,k}(x)$ **best approximation** (least error):

$$\|f - S_{f,k}\|^2 \leq \|f - P_k\|^2 \quad \text{for all } P_k(x)$$

$$\text{Obs: } \|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^k (a_n^2 + b_n^2) \right]$$

- ③ **Parseval's identity**

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx \quad \left(\text{when } \int_{-\pi}^{\pi} f^2 dx < \infty \right)$$

Summary: Fourier series

- 5 Real Fourier series of $2L$ -periodic f :

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

- 6 Complex Fourier series of $2L$ -periodic f :

$$S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}}, \quad c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-i \frac{n\pi x}{L}} dx$$

$$\text{OBS: } \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}} = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) !!!$$

Lecture 8: Fourier integrals and transforms

Kreyszig: Section 11.7, 11.9

- 1 Fourier integral
- 2 Fourier transform
- 3 Properties
- 4 Examples

Homework: Repeat complex *numbers*, *absolute values*, *exponentials* [Mat 3]

Lecture 8: Fourier integral

Fourier integral of $f(x)$, $x \in \mathbb{R}$:

$$I_f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw,$$

where

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx.$$

Lecture 8: Fourier transform

Fourier transform of $f(x)$, $x \in \mathbb{R}$:

$$\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Invers:

$$\mathcal{F}^{-1}[g](x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{iwx} dw$$

Lecture 8: Fourier transform

Properties:

$$\mathcal{F}[af(x) + bg(x)](w) = a\mathcal{F}[f](w) + b\mathcal{F}[g](w)$$

$$\mathcal{F}[f'](w) = (iw)\mathcal{F}[f](w) \quad (|f(x)| \xrightarrow{|x| \rightarrow \infty} 0)$$

$$\mathcal{F}[e^{-iax}f(x)](w) = \mathcal{F}[f](w + a)$$

Summary: Fourier integral and transform

- ① **Fourier integral** of $f(x)$, $x \in \mathbb{R}$:

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- ② **Fourier transform** of $f(x)$, $x \in \mathbb{R}$:

$$\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Invers:
$$\mathcal{F}^{-1}[g](x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{iwx} dw$$

Under certain conditions: $f(x) = I_f(x) = \mathcal{F}^{-1}[\mathcal{F}[f]](x)$

- ③ **Properties:**

$$\mathcal{F}[af(x) + bg(x)](w) = a\mathcal{F}[f](w) + b\mathcal{F}[g](w)$$

$$\mathcal{F}[f'](w) = (iw)\mathcal{F}[f](w) \quad (|f(x)| \rightarrow 0 \text{ as } |x| \rightarrow \infty)$$

$$\mathcal{F}[e^{-iax} f(x)](w) = \mathcal{F}[f](w + a)$$