

Summary: Fourier integral and transform

- ① **Fourier integral** of $f(x)$, $x \in \mathbb{R}$:

$$I_f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

- ② **Fourier transform** of $f(x)$, $x \in \mathbb{R}$:

$$\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Invers: $\mathcal{F}^{-1}[g](x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{iwx} dw$

Under certain conditions: $f(x) = I_f(x) = \mathcal{F}^{-1}[\mathcal{F}[f]](x)$

- ③ **Properties:**

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$$

$$\mathcal{F}^{-1}[a\hat{f} + b\hat{g}] = af + bg$$

$$\mathcal{F}[f'](w) = (iw)\mathcal{F}[f](w)$$

$$\mathcal{F}^{-1}[\hat{f}'(w)](x) = (-ix)f(x)$$

$$\mathcal{F}[e^{-iax}f(x)](w) = \mathcal{F}[f](w + a)$$

$$\mathcal{F}^{-1}[e^{iaw}\hat{f}(w)](x) = f(x + a)$$

Lecture 9: Fourier transform and partial differential equations

Kreyszig: Section 11.9, 12.1

- 1 Fourier transform
- 2 Convolution
- 3 Introduction to partial differential equations
- 4 Examples

Homework:

- 1 Read Kreyszig 12.2 yourselves.
- 2 Repeat complex *numbers, absolute values, exponentials*
- 3 Repeat *Solution of ordinary differential equations* [Mat 3]

Lecture 9: Fourier Transform

Important transform:

$$\mathcal{F}[e^{-ax^2}](w) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$$

Remember: $\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$$

$$\mathcal{F}^{-1}[a\hat{f} + b\hat{g}] = af + bg$$

$$\mathcal{F}[f'](w) = (iw)\mathcal{F}[f](w)$$

$$\mathcal{F}^{-1}[\hat{f}'(w)](x) = (-ix)f(x)$$

$$\mathcal{F}[e^{-iax} f(x)](w) = \mathcal{F}[f](w + a)$$

$$\mathcal{F}^{-1}[e^{iaw} \hat{f}(w)](x) = f(x + a)$$

Lecture 9: Fourier Transform

Convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$$

$$\boxed{\mathcal{F}[f * g](w) = \sqrt{2\pi} \mathcal{F}[f](w) \cdot \mathcal{F}[g](w)}$$

Remember:

$$\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx$$

Partial differential equations (PDEs)

Example:
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (\text{or} \quad u_t = c^2 u_{xx})$$

Order, linear, homogeneous, solution

Linear + homogeneous \Rightarrow Superposition

Uniqueness: Need boundary and initial conditions

Lecture 9: PDEs

2nd order PDEs

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

Type	Condition	Typical example
Hyperbolic	$AC - B^2 < 0$	$u_{tt} - u_{xx} = 0$ (<i>wave eqn</i>)
Parabolice	$AC - B^2 = 0$	$u_t - u_{xx} = 0$ (<i>heat eqn</i>)
Elliptic	$AC - B^2 > 0$	$u_{xx} + u_{yy} = 0$ (<i>Laplace eqn</i>)

Summary: Fourier transform and PDEs

① **Fourier transform** $\mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$

$$\mathcal{F}[e^{-ax^2}](w) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$$

$$\mathcal{F}[f * g](w) = \sqrt{2\pi} \mathcal{F}[f](w) \cdot \mathcal{F}[g](w) \quad f * g(x) = \int_{-\infty}^{\infty} f(y) g(x - y) dy$$

② **Partial differential equations (PDEs)**

Equations involving partial derivatives of the unknown

Concepts: Order, linear, homogeneous, hyperbolic/parabolic/elliptic

Solution: u solution of PDE in region R if

(i) all derivatives appearing in PDE exist and are continuous in R

(ii) u satisfy the PDE in all points in R

Superposition/Linearity:

u_1 and u_2 solve same *linear, homogeneous* PDE in R ; $a, b \in \mathbb{R}$

$\Rightarrow au_1 + bu_2$ solves same PDE in R

Unique solution:

Need also **boundary** and **initial conditions**!