

Summary: Fourier transform and PDEs

① **Fourier transform** $\mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$

$$\mathcal{F}[e^{-ax^2}](w) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$$

$$\mathcal{F}[f * g](w) = \sqrt{2\pi} \mathcal{F}[f](w) \cdot \mathcal{F}[g](w) \quad f * g(x) = \int_{-\infty}^{\infty} f(y) g(x - y) dy$$

② **Partial differential equations (PDEs)**

Equations involving partial derivatives of the unknown

Concepts: Order, linear, homogeneous, hyperbolic/parabolic/elliptic

Solution: u solution of PDE in region R if

(i) all derivatives appearing in PDE exist and are continuous in R

(ii) u satisfy the PDE in all points in R

Superposition/Linearity:

u_1 and u_2 solve same *linear, homogeneous* PDE in R ; $a, b \in \mathbb{R}$

$\Rightarrow au_1 + bu_2$ solves same PDE in R

Unique solution:

Need also **boundary** and **initial conditions!**

Lecture 10: Partial differential equations (PDEs)

Kreyszig: Section 12.3

- ➊ Introduction to partial differential equations (continued)
- ➋ Solution technique: Separation of variables
- ➌ Example PDE: The wave equation

Homework:

- ➊ Read Kreyszig 12.2 yourselves.
- ➋ Repeat *Solution of ordinary differential equations* [Mat 3]

Lecture 10: Separation of variables

$$(1) \quad u_{tt} = c^2 u_{xx} \quad t > 0, x \in (0, L)$$

$$(2) \quad u(0, t) = 0 = u(L, t) \quad t > 0, x \in \{0, L\}$$

$$(3) \quad u(x, 0) = f(x) \quad t = 0, x \in (0, L)$$

$$(4) \quad u_t(x, 0) = g(x) \quad t = 0, x \in (0, L)$$

1. Product solutions $u(x, t) = F(x)G(t)$

(a) Derive ODEs and conditions for F and G , from (1) and homogeneous conditions (2):

$$F'' - kF = 0$$

$$G'' - c^2 kG = 0 \quad (k = \text{constant})$$

$$F(0) = 0 = F(L)$$

Lecture 10: Separation of variables

- (1) $u_{tt} = c^2 u_{xx}$ $t > 0, x \in (0, L)$
- (2) $u(0, t) = 0 = u(L, t)$ $t > 0, x \in \{0, L\}$
- (3) $u(x, 0) = f(x)$ $t = 0, x \in (0, L)$
- (4) $u_t(x, 0) = g(x)$ $t = 0, x \in (0, L)$

1. Product solutions $u(x, t) = F(x)G(t)$

(a) Derive ODEs and hom. cond'ns for F and G .

(b) Solve for F : $F_n(x) = \sin \frac{n\pi x}{L}$

(c) Solve for G : $G_n(t) = B_n \cos \frac{cn\pi t}{L} + B_n^* \sin \frac{cn\pi t}{L}$

(d) All product solution:

$$u_n(x, t) = F_n(x)G_n(t), n \in \{0, 1, 2, 3, \dots\}$$

Lecture 10: Separation of variables

2. Superposition and (3) and (4)

Solution candidate:

$$u(x, t) = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \left(B_n \cos \frac{cn\pi t}{L} + B_n^* \sin \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L}$$

Satisfy nonhomogeneous conditions (3) and (4):

$$f(x) \stackrel{(3)}{=} u(x, 0) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi x}{L}$$

$$\Rightarrow_{F-\sin \text{ series}} \boxed{B_n = \frac{2}{L} \int_0^L f \sin \frac{n\pi x}{L} dx}$$

$$g(x) \stackrel{(4)}{=} u_t(x, 0) = \sum_{n=0}^{\infty} B_n^* \frac{cn\pi}{L} \sin \frac{n\pi x}{L}$$

$$\Rightarrow \boxed{B_n^* \frac{cn\pi}{L} = \frac{2}{L} \int_0^L g \sin \frac{n\pi x}{L} dx}$$

Separation of variables – wave equation

$$(1) \quad u_{tt} = c^2 u_{xx} \quad t > 0, x \in (0, L)$$

$$(2) \quad u(0, t) = 0 = u(L, t) \quad t > 0, x \in \{0, L\}$$

$$(3) \quad u(x, 0) = f(x) \quad t = 0, x \in (0, L)$$

$$(4) \quad u_t(x, 0) = g(x) \quad t = 0, x \in (0, L)$$

1 Separation of variables $u(x, t) = F(x)G(t)$

$$(1) \text{ and } (2) \Rightarrow \boxed{F'' - kF = 0} \quad \boxed{G'' - c^2 k G = 0} \quad \boxed{F(0) = 0 = F(L)}$$

2 Find *all* $u = FG$ solutions of (1) and (2) [linear, homogeneous]

Only $u \neq 0$ if $k = -\left(\frac{n\pi}{L}\right)^2$

$$\boxed{F_n(x) = \sin \frac{n\pi x}{L}} \quad \boxed{G_n(t) = B_n \cos \frac{cn\pi t}{L} + B_n^* \sin \frac{cn\pi t}{L}}$$

$$\boxed{u_n(x, t) = F_n(x)G_n(t)} \quad n \in \{0, 1, 2, 3, \dots\}$$

Separation of variables – wave equation

$$(1) \quad u_{tt} = c^2 u_{xx} \quad t > 0, x \in (0, L)$$

$$(2) \quad u(0, t) = 0 = u(L, t) \quad t > 0, x \in \{0, L\}$$

$$(3) \quad u(x, 0) = f(x) \quad t = 0, x \in (0, L)$$

$$(4) \quad u_t(x, 0) = g(x) \quad t = 0, x \in (0, L)$$

4 Superposition and (3) and (4)

[inhomogeneous cond'ns]

$$u(x, t) = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \left(B_n \cos \frac{cn\pi t}{L} + B_n^* \sin \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L}$$

$$f(x) \stackrel{(3)}{=} u(x, 0) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi x}{L} \quad \begin{array}{l} F\text{-sin series} \\ \Rightarrow \end{array} \quad B_n = \frac{2}{L} \int_0^L f \sin \frac{n\pi x}{L} dx$$

$$g(x) \stackrel{(4)}{=} u_t(x, 0) = \sum_{n=0}^{\infty} B_n^* \frac{cn\pi}{L} \sin \frac{n\pi x}{L} \Rightarrow B_n^* \frac{cn\pi}{L} = \frac{2}{L} \int_0^L g \sin \frac{n\pi x}{L} dx$$

u, u_t sinus series at $t = 0 \Rightarrow$ use Fourier sinus series of f, g

u solution if series converges and 2x term-wise differentiation ok