

Summary: Separation of variables – wave equation

$$\begin{aligned}(1) \quad & u_{tt} = c^2 u_{xx} & t > 0, x \in (0, L) \\(2) \quad & u(0, t) = 0 = u(L, t) & t > 0, x = 0, L \\(3) \quad & u(x, 0) = f(x), u_t(x, 0) = g(x) & t = 0, x \in (0, L)\end{aligned}$$

① Find all solutions $u(x, t) = F(x)G(t)$ of (1) and hom. cond'ns (2):

$$\xRightarrow{(1),(2)} \boxed{F'' - kF = 0} \quad \boxed{G'' - c^2 kG = 0} \quad \boxed{F(0) = 0 = F(L)}$$

$$\xRightarrow{u \neq 0} \boxed{k = -\left(\frac{n\pi}{L}\right)^2} \quad \boxed{F_n(x) = \sin \frac{n\pi x}{L}} \quad \boxed{G_n(t) = B_n \cos \frac{cn\pi t}{L} + B_n^* \sin \frac{cn\pi t}{L}}$$

② Superposition and inhomogeneous conditions (3):

$$\boxed{u(x, t) = \sum_{n=0}^{\infty} F_n G_n = \sum_{n=0}^{\infty} \left(B_n \cos \frac{cn\pi t}{L} + B_n^* \sin \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L}}$$

$$f(x) \stackrel{(3)}{=} u(x, 0) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi x}{L} \quad \xRightarrow{F-\sin \text{ series}} \boxed{B_n = \frac{2}{L} \int_0^L f \sin \frac{n\pi x}{L} dx}$$

$$g(x) \stackrel{(3)}{=} u_t(x, 0) = \sum_{n=0}^{\infty} B_n^* \frac{cn\pi}{L} \sin \frac{n\pi x}{L} \Rightarrow \boxed{B_n^* \frac{cn\pi}{L} = \frac{2}{L} \int_0^L g \sin \frac{n\pi x}{L} dx}$$

Lecture 11: Partial differential equations

Kreyszig: Section 12.6

- 1 PDEs: Heat equation, Laplace equation
- 2 Boundary value problems: Cauchy, Dirichlet, Neumann.
- 3 Solution technique: Separation of variables

Homework:

- 1 Repeat *Solution of ordinary differential equations* [Mat 3]
See slides on wiki!

Lecture 11: Separation of var's – heat equation

$$(1) \quad u_t = c^2 u_{xx} \quad t > 0, x \in (0, L)$$

$$(2) \quad u(0, t) = 0 = u(L, t) \quad t > 0, x = 0, L$$

$$(3) \quad u(x, 0) = f(x) \quad t = 0, x \in (0, L)$$

1. Find all solutions $u(x, t) = F(x)G(t)$ of (1) and (2):

$$\Rightarrow_{(1),(2)} \boxed{F'' - kF = 0} \quad \boxed{G' - c^2 kG = 0} \quad \boxed{F(0) = 0 = F(L)}$$

$$\Rightarrow_{u \neq 0} \boxed{k = -\left(\frac{n\pi}{L}\right)^2} \quad \boxed{F_n(x) = \sin \frac{n\pi x}{L}} \quad \boxed{G_n(t) = B_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}}$$

2. Superposition and inhomogeneous condition (3):

$$\boxed{u(x, t) = \sum_{n=0}^{\infty} F_n G_n} \quad \dots \text{ use (3) and Fourier sin series } \dots$$

Lecture 11: Separation of var's – Laplace equation

$$(1) \quad u_{xx} + u_{yy} = 0 \quad x \in (0, a), \quad y \in (0, b)$$

$$(2) \quad u(0, y) = 0 = u(a, y) \quad y \in [0, b]$$

$$(3) \quad u(x, 0) = 0, u(x, b) = f(x) \quad x \in [0, a]$$

1. Find all solutions $u(x, y) = F(x)G(y)$ of (1), (2), and (3)₁:

$$\Rightarrow F'' - kF = 0, G'' + kG = 0, F(0) = 0 = F(a), G(0) = 0$$

$$\Rightarrow \boxed{k = -\left(\frac{n\pi}{a}\right)^2} \quad \boxed{F_n(x) = \sin \frac{n\pi x}{a}} \quad \boxed{G_n(y) = 2A_n \sinh \frac{n\pi y}{a}}$$

2. Superposition and inhomogeneous condition (3)₂:

$$\boxed{u(x, y) = \sum_{n=0}^{\infty} F_n G_n} \quad \dots \text{ use (3)}_2 \text{ and Fourier sin series } \dots$$

Summary: Heat and Laplace equation

1 Boundary value problems:

Cauchy u given at $t = 0$.

Dirichlet u given on boundary

Neumann (normal) derivative of u given on boundary

2 Heat equation:

$$(1) \quad u_t = c^2 u_{xx} \quad t > 0, x \in (0, L)$$

$$(2) \quad u(0, t) = 0 = u(L, t) \quad t > 0, x = 0, L$$

$$(2') \quad u_x(0, t) = 0 = u_x(L, t) \quad t > 0, x = 0, L$$

$$(3) \quad u(x, 0) = f(x) \quad t = 0, x \in (0, L)$$

u temperature of rod, ends: fixed temperature (2) or insulated (2')

Cauchy-Dirichlet (1), (2), (3); Cauchy-Neumann (1), (2'), (3)

3 Laplace equation:

$$(4) \quad u_{xx} + u_{yy} = 0$$

Electrostatic potential, potential flow, membrane, temperature ...

Summary: Separation of variables – heat equation

$$(1) \quad u_t = c^2 u_{xx} \quad t > 0, x \in (0, L)$$

$$(2') \quad u_x(0, t) = 0 = u_x(L, t) \quad t > 0, x = 0, L$$

$$(3) \quad u(x, 0) = f(x) \quad t = 0, x \in (0, L)$$

❶ Find all solutions $u(x, t) = F(x)G(t)$ of (1) and hom. cond'ns (2'):

$$\begin{array}{l} \Rightarrow \\ (1), (2') \end{array} \quad \boxed{F'' - kF = 0} \quad \boxed{G' - c^2 kG = 0} \quad \boxed{F'(0) = 0 = F'(L)}$$

$$\begin{array}{l} \Rightarrow \\ u \neq 0 \end{array} \quad \boxed{k = -\left(\frac{n\pi}{L}\right)^2} \quad \boxed{F_n(x) = \cos \frac{n\pi x}{L}} \quad \boxed{G_n(t) = B_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}}$$

❷ Superposition and inhomogeneous condition (3):

$$u(x, t) = \sum_{n=0}^{\infty} F_n G_n = \sum_{n=0}^{\infty} B_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \cos \frac{n\pi x}{L}$$

$$f(x) \stackrel{(3)}{=} u(x, 0) = \sum_{n=0}^{\infty} B_n \cos \frac{n\pi x}{L}$$

$$\begin{array}{l} \Rightarrow \\ F-\cos \text{ series} \end{array} \quad \boxed{B_0 = \frac{1}{L} \int_0^L f \, dx} \quad \boxed{B_n = \frac{2}{L} \int_0^L f \cos \frac{n\pi x}{L} \, dx}$$