

# Summary Lecture 17: Complex Analysis

## 1 Logarithm:

$$\ln z = \ln |z| + i(\operatorname{Arg} z + 2\pi n), \quad n \in \mathbb{Z}$$

$$\operatorname{Ln} z = \ln |z| + i\operatorname{Arg} z \quad (\text{principal value})$$

$$(\operatorname{Ln} z)' = \frac{1}{z} \quad \boxed{\operatorname{Ln} z \text{ analytic}} \text{ except at } z = 0 \text{ and negative real axis}$$

$$z^c \stackrel{\text{DEF}}{=} e^{c \ln z}$$

## 2 Complex line integral:

Defined via Riemann-sums, exists and uniquely defined when:

(A1)  $C$  piecewise smooth, oriented curve with finite length

(A2)  $f$  is continuous on  $C$

$$\int_C f(z) dz = \int_a^b f(z(t)) \dot{z}(t) dt \quad \text{when } C : z(t), t \in [a, b]$$

$$\int_C [af(z) + bg(z)] dz = a \int_C f(z) dz + b \int_C g(z) dz$$

$$\int_{-C} f(z) dz = - \int_C f(z) dz$$

$$\int_{C_1 \cup C_2} f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz \quad \text{when } C_1 \cap C_2 = \emptyset$$

# Lecture 18: Complex Analysis

Kreyszig: Sections 14.1, 14.2

- 1 Complex Line integral
- 2 Cauchy integral theorem
- 3 Independence of path
- 4 The indefinite integral
- 5 Domains with holes
- 6 Examples

## Homework:

Repeat by next week: Sequences, series, convergence tests  
[Mat 1/GKA 2]

## Lecture 18: Complex line integral

$$\int_C [af(z) + bg(z)]dz = a \int_C f(z)dz + b \int_C g(z)dz$$

$$\int_{-C} f(z)dz = - \int_C f(z)dz$$

$$\int_{C_1 \cup C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \text{ when } C_1 \cap C_2 = \emptyset$$

$$\left| \int_C f(z)dz \right| \leq M \cdot L$$

$$\text{where } M = \max_{z \in C} |f(z)|, \quad L = \text{length}(C)$$

## Lecture 18: Cauchy's integral theorem

$f$  analytic in simply connected domain  $D$ ,  $C \subset D$   
simple, closed curve

$$\implies \oint_C f(z) dz = 0$$

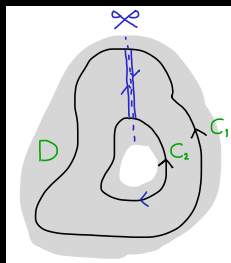
Consequences:

- (a)  $\int_C f(z) dz$  is independent of path in  $D$   
(depends on endpoints of  $C$  only)
- (b) The indefinite integral of  $f$  exists in  $D$

## Lecture 18: Domains with holes

Cut to have a simply connected domain...

... add segments along cut to have closed curve...



Cauchy in the cut domain  $D^*$ :

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$

# Summary Lecture 18: Complex Analysis

## 1 Complex line integral:

$$\int_C [af(z) + bg(z)]dz = a \int_C f(z)dz + b \int_C g(z)dz$$

$$\int_{-C} f(z)dz = - \int_C f(z)dz$$

$$\int_{C_1 \cup C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \quad \text{when} \quad C_1 \cap C_2 = \emptyset$$

$$\boxed{|\int_C f(z)dz| \leq M \cdot L} \quad M = \max_{z \in C} |f(z)|, \quad L = \text{length of } C$$

## 2 Cauchy's integral theorem

$f$  analytic in simply connected domain  $D$ ,  $C \subset D$  simple, closed curve

$$\implies \oint_C f(z)dz = 0$$

## 3 Corollaries:

(a)  $\int_C f(z)dz$  is independent of path in  $D$

(b) The indefinite integral of  $f$  exists in  $D$

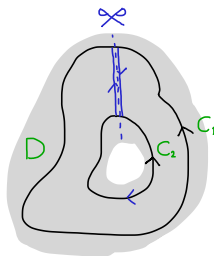
# Summary Lecture 18: Complex Analysis

## 4 Domains with holes:

Cut to have a simply connected domain...

... add segments along cut to have closed curve...

Then use Cauchy:



Cauchy in the cut domain  $D^*$  and cancelations along cut:

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$