

Summary: Cauchy's integral formula ...

1 Cauchy's integral formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \quad \text{if}$$

(A1) f is analytic in simply connected domain D

(A2) $z_0 \in D$, $C \subset D$ simple closed curve, positively oriented, enclosing z_0 .

2 Infinitely differentiable:

f analytic in $D \Rightarrow f$ infinitely differentiable in D , and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

3 Properties of analytic functions:

Cauchy's inequality: f analytic $\Rightarrow |f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|$

Liouville's theorem: f analytic, bounded in $\mathbb{C} \Rightarrow f$ is constant

Morera's theorem: f continuous in simply connected domain D and $\oint_C f(z) dz = 0$ for all simple, closed $C \subset D \Rightarrow f$ analytic in D

Lecture 20: Complex power series

Kreyszig: Sections 15.1, 15.2

- 1 Complex sequences and series
- 2 Complex power series
- 3 Convergence and divergence
- 4 Radius of convergence
- 5 Examples

Homework:

Repeat by next week [Mat 1/GKA 2]:

Taylor series. How to find/work with them, Taylor's thm, examples

Lecture 20: Complex series and sequences

Convergence (absolute/not), divergence, Cauchy

Geometric series

Convergence tests:

Comparison, ratio, root and divergence tests

Lecture 20: Complex power series

$$(1) \sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

Center z_0 , coefficients a_n $[(z - z_0)^0 = 1]$

Convergence in z_1

$\Rightarrow$ convergence in z for all $|z - z_0| < |z_1 - z_0|$

Divergence in z_2

$\Rightarrow$ divergence in z for all $|z - z_0| > |z_2 - z_0|$

Lecture 20: Radius of convergence

Distance $R = |z_0 - z^*|$ to nearest point z^* where power series diverges

- Exists always
- Series converges (diverges) if $|z - z_0| < R$ ($> R$)

Cauchy-Hadamard:

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \quad (\text{when the limit exists})$$

Summary Lecture 20: Complex power series

1 Complex series and sequences:

Definitions, results and proofs – similar to real case

Convergence, absolute convergence, divergence

Comparison, ratio, and root test; divergence test; geometric series

2 Complex power series:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots$$

Center z_0 , coefficients a_n , $(z - z_0)^0 = 1$ by definition

Convergence in $z_1 \Rightarrow$ convergence in z for all $|z - z_0| < |z_1 - z_0|$

Divergence in $z_2 \Rightarrow$ divergence in z for all $|z - z_0| > |z_2 - z_0|$

3 Radius of convergence R :

Distance $R = |z_0 - z^*|$ to nearest point z^* where power series diverges

Always exists; series converges (diverges) if $|z - z_0| < R$ ($> R$)

Cauchy-Hadamard: $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$ when the limit exists