

Summary: Complex (power) series

1 Complex series and sequences:

Definitions, results and proofs – similar to real case

Convergence, absolute convergence, divergence

Comparison, ratio, root, and divergence tests; geometric series

2 Complex power series:

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

Center z_0 , coefficients a_n , $(z - z_0)^0 = 1$ by definition

Convergence in $z_1 \Rightarrow$ convergence in z for all $|z - z_0| < |z_1 - z_0|$

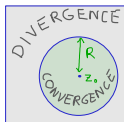
Divergence in $z_2 \Rightarrow$ divergence in z for all $|z - z_0| > |z_2 - z_0|$

3 Radius of convergence R :

Distance $R = |z_0 - z^*|$ to nearest point z^* where power series diverges

Always exists; series converges (diverges) if $|z - z_0| < R$ ($> R$)

Cauchy-Hadamard: $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$ when the limit exists



Lecture 21: Complex power series

Kreyszig: Sections 15.3, 15.4

- 1 Term wise operations on power series
- 2 Power series representation of functions
- 3 Taylor's formula, Taylor series, definition and convergence
- 4 Examples

Homework:

Repeat if necessary [Mat 1/GKA 2]:

Taylor series. How to find/work with them, Taylor's thm, examples

Lecture 21: Termwise operations on power series

$$c_1 \sum_{n=0}^{\infty} a_n z^n + c_2 \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n) z^n$$
$$[R = \min(R_a, R_b)]$$

$$\left(\sum_{n=0}^{\infty} a_n z^n \right)' = \sum_{n=1}^{\infty} n a_n z^{n-1}$$
$$[R = R_a]$$

$$\int \left(\sum_{n=0}^{\infty} a_n z^n \right) dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} + C$$
$$[R = R_a]$$

Lecture 21: Represent functions by power series

Represents their sum where they converges

Uniqueness:

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n \quad \Rightarrow \quad a_n = b_n, \quad n = 0, 1, 2, \dots$$

The sum is analytic where the power series converges

Lecture 21: Taylor's formula, Taylor series

Taylor series:
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

Taylor's formula:
$$f(z) = \sum_{n=0}^m a_n (z - z_0)^n + R_m(z, z_0)$$

Taylor's theorem:

f analytic in domain D and $z_0 \in D$

$\implies$ the Taylor series of f about z_0 **exists**, is **unique**, and **converges**
to f in the largest open disc about z_0 where f is analytic

Summary Lecture 21: Complex power series

1 Termwise operations on power series:

$$(a) \quad c_1 \sum_{n=0}^{\infty} a_n z^n + c_2 \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n) z^n \quad [R = \min(R_a, R_b)]$$

$$(b) \quad \left(\sum_{n=0}^{\infty} a_n z^n \right)' = \sum_{n=1}^{\infty} n a_n z^{n-1} \quad [R = R_a]$$

$$(c) \quad \int \left(\sum_{n=0}^{\infty} a_n z^n \right) dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} + C \quad [R = R_a]$$

2 Representation of functions by power series:

Represents their sum where they converges

$$\text{Uniqueness: } \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n \Rightarrow a_n = b_n, \quad n = 0, 1, 2, \dots$$

The function is analytic where the power series converges

Summary Lecture 21: Complex power series

3 Taylor series:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

4 Taylor's theorem:

If f is analytic in a domain D and $z_0 \in D$, then

- (a) the Taylor series of f about z_0 exists and is unique,
- (b) and it converges and equals f in the largest open disc about z_0 where f is analytic

