

Summary: Complex power series

1 Termwise operations on power series:

$$(a) \quad c_1 \sum_{n=0}^{\infty} a_n z^n + c_2 \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n) z^n \quad [R = \min(R_a, R_b)]$$

$$(b) \quad \left(\sum_{n=0}^{\infty} a_n z^n \right)' = \sum_{n=1}^{\infty} n a_n z^{n-1} \quad [R = R_a]$$

$$(c) \quad \int \left(\sum_{n=0}^{\infty} a_n z^n \right) dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} + C \quad [R = R_a]$$

2 Representation of functions by power series:

Represents their sum where they converges

$$\text{Uniqueness: } \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n \Rightarrow a_n = b_n, \quad n = 0, 1, 2, \dots$$

The function is analytic where the power series converges

Summary Lecture 21: Complex power series

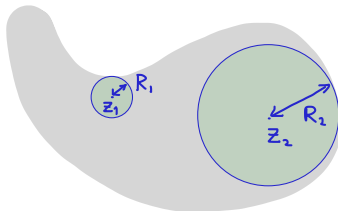
3 Taylor series:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

4 Taylor's theorem:

If f is analytic in a domain D and $z_0 \in D$, then

- (a) the Taylor series of f about z_0 exists and is unique,
- (b) and it converges and equals f in the largest open disc about z_0 where f is analytic



Lecture 22: Taylor and Laurent series

Kreyszig: Sections 15.4, 16.1

- 1 Important power series
- 2 Finding power series
- 3 Laurent series, definition and convergence
- 4 Examples

Homework:

Repeat *Taylor series* [Mat 1/GKA 1]

Lecture 22: Taylor series

Important series:

$$e^z, \cos z, \sin z, \cosh z, \dots, \operatorname{Ln} z, \dots, \frac{1}{1-z}, \dots$$

Finding Taylor series: (as for real series)

Use known series, term wise operations
(addition, derivation, integration),

substitution, geometric series, binomial series,
partial fractions.

Lecture 22: Laurent series

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Example:

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots$$

Lecture 22: Laurent's theorem

(a) There **exists** a **unique** Laurent series (1) that **converges to f** in the **largest annulus**

$$D : r < |z - z_0| < R$$

where f is analytic.

$$(b) \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*, \quad b_n = a_{-n}.$$

Summary Lecture 22: Taylor and Laurent series

1 Laurent series:

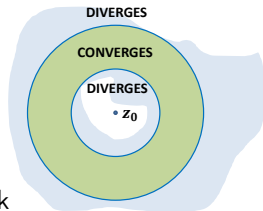
$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$\text{Example: } e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots$$

2 Laurent's theorem:

(a) There exists a unique Laurent series (1) that converges to f in the largest annulus $D : r < |z - z_0| < R$ where f is analytic.

$$(b) \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*,$$
$$b_n = \frac{1}{2\pi i} \oint_C f(z^*) (z^* - z_0)^{n-1} dz^*$$



3 Remarks:

Term wise addition, differentiation, ... is ok

a_n, b_n found from known Taylor series, substitution, ...