

Summary: Taylor and Laurent Series

1 Laurent series:

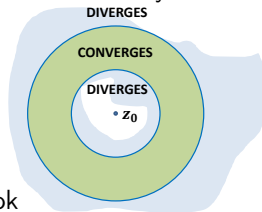
$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$\text{Example: } e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots$$

2 Laurent's theorem:

(a) There exists a unique Laurent series (1) that converges to f in the largest annulus $D : r < |z - z_0| < R$ where f is analytic.

$$(b) \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*,$$
$$b_n = a_{-n}.$$



3 Remarks:

Term wise addition, differentiation, ... is ok

a_n, b_n found from known Taylor series, substitution, ...

Lecture 23: Singularities and Residue integration

Kreyszig: Sections 16.2, 16.3

- 1 Singular points
- 2 Zeros
- 3 Classification, properties, $z = \infty$
- 4 Residue integration
- 5 Examples

Lecture 23: Singularities

Points z_0 where $f(z)$ is not analytic/defined

Isolated singularity z_0 , the only one in some nbhd

$\Downarrow$ Laurent's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\neq 0}, \quad 0 < |z - z_0| < R$$

Principal value of z_0 :

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

z_0 order n pole:

$$b_n \neq 0, \quad b_k = 0, \quad k > n$$

z_0 isolated essential singularity: $b_n \neq 0$ for ∞ many n

Lecture 23: Zeros and infinity

Zeros of order m

Zeros are isolated; f has zero $\implies 1/f$ has pole

$f(z)$ has pole/essential singularity/zero at $z = \infty$

$\Updownarrow$ DEF

$f(\frac{1}{w})$ has pole/essential singularity/zero at $w = 0$

Lecture 23: Residue integration

z_0 only singularity of $f(z)$ enclosed by C

$\Downarrow$ Laurent's theorem

$$f(z) = a_0 + a_1(z - z_0) + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots,$$
$$(0 < |z - z_0| < R)$$

$\Downarrow$ Term wise integration/Laurent's theorem

$$\oint_C f(z) dz = 2\pi i b_1$$

Residue of $f(z)$ at z_0 : $\operatorname{Res}_{z=z_0} f(z) = b_1$

Summary Lecture 23: Singularities

❶ **Singularity:** Point z_0 where $f(z)$ is not analytic/defined (...)

❷ **Isolated singularity:**

z_0 is the only singularity in $|z - z_0| < R$ for some $R > 0$

⇓ **Laurent's theorem**

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\neq 0}, \quad 0 < |z - z_0| < R$$

Principal value of z_0 :

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

z_0 order n pole:

$$b_n \neq 0, b_k = 0, k > n$$

z_0 isolated essential singularity:

$$b_n \neq 0 \text{ for } \infty \text{ many } n$$

Summary Lecture 23: Residue integration

3 Residue integration:

$$\text{Goal: } \oint_C f(z) dz = 2\pi i \sum \text{residues}$$

4 Only one singularity:

z_0 only singularity of $f(z)$ enclosed by C (simple, closed, counter cl.wise)

$\Downarrow$ Laurent's theorem

$$f(z) = a_0 + a_1(z - z_0) + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots, \quad 0 < |z - z_0| < R$$

$\Downarrow$ Term wise integration/Laurent's theorem

$$\oint_C f(z) dz = 2\pi i b_1$$

5 Residue of $f(z)$ at z_0 :

$$\operatorname{Res}_{z=z_0} f(z) = b_1$$

b_1 -coefficient in Laurent series that converges in $0 < |z - z_0| < R$!