

Summary: Residue integration

- ① **Residues:** $\operatorname{Res}_{z=z_0} f(z) = b_1$ where

$$f(z) = \sum a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots, \quad 0 < |z-z_0| < R$$

z_0 order 1 pole:

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} (z-z_0)f(z)$$

or

$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

z_0 order n pole:

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \left((z-z_0)^n f(z) \right)^{(n-1)}$$

z_0 isolated essential singularity:

Find b_1 from the Laurent series!

- ② **Residue theorem:** Assume

(A1) C simple, closed curve oriented counterclockwisely

(A2) $f(z)$ has finite number of singularities $z_1, \dots, z_m$ enclosed by C

Then

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^m \operatorname{Res}_{z=z_j} f(z)$$

Lecture 25: Residue integration of real integrals

Kreyszig: Section 16.4

1 Computing real integrals:

Type I: $\int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta$

Type II: $\int_{-\infty}^{\infty} f(x) dx$

Type III: $\int_{-\infty}^{\infty} f(x)e^{-iwx} dx$

2 Examples

Information (check [www](#)):

Øving 13, 8/13 øvinger for å ta eksamen, neste uke

Frist øving 12 og 13:

Denne uka! Fredag kl 12:00!

Lecture 25: Type I real integrals

$$I = \int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta, \quad F \text{ rational}$$

Substitution:

$$z = e^{i\theta}$$

$$\cos \theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin \theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$dz = iz d\theta$$

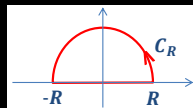
$$\text{Interval } [0, 2\pi) \rightsquigarrow \text{circle } |z| = 1$$

Lecture 25: Type II real integrals

$$I = \int_{-\infty}^{\infty} f(x) dx, \quad f \text{ rational, no real singularities}$$

$$(i) \quad I = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$$

$$(ii) \quad \int_{-R}^R f(x) dx = \oint_{C_R} f(z) dz - \int_{S_R} f(z) dz$$



(iii) R big enough $\Rightarrow C_R$ encircles all poles in upper half plane

$$\oint_{C_R} f(z) dz \stackrel{\text{Residue thm.}}{=} 2\pi i \sum_{\text{poles in upper half plane}} \text{Res } f(z)$$

$$(iv) \quad \left| \int_{S_R} f(z) dz \right| \leq \max_{z \in S_R} |f(z)| \cdot L \leq \max_{|z|=R} |f(z)| \cdot \pi R \xrightarrow[R \rightarrow \infty]{\text{Must show}} 0$$

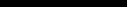
Lecture 25: Type III real integrals

$$I = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx, \quad f \text{ rational, no real sing.}$$

Same procedure as for Type II integrals...

... and choose C_R to insure $w \cdot \text{Im } z \leq 0$:

$w \leq 0 \rightsquigarrow$

;
 $w > 0 \rightsquigarrow$


$$\Rightarrow |f(z)e^{-iwz}| \stackrel{z=x+iy}{=} |f(z)|e^{w \cdot y} \stackrel{z \in C_R}{\leq} |f(z)|$$

$\operatorname{Re} I = \int_{-\infty}^{\infty} f(x) \cos(wx) dx$
 $\operatorname{Im} I = - \int_{-\infty}^{\infty} f(x) \sin(wx) dx$

Summary Lecture 25: Real integrals

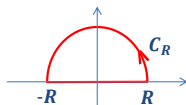
① Type I: $I = \int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta$ F rational

Substitution: $z = e^{i\theta}$, $\cos \theta = \frac{1}{2}(z + \frac{1}{z})$, $\dots$, $[0, 2\pi) \rightsquigarrow |z| = 1$

② Type II: $I = \int_{-\infty}^{\infty} f(x) dx$ f rational, no real singularities, $\dots$

(i) $I = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$

(ii) $\int_{-R}^R f(x) dx = \oint_{C_R} f(z) dz - \int_{S_R} f(z) dz$



(iii) R big enough $\Rightarrow C_R$ encircles all poles in upper half plane

$$\oint_{C_R} f(z) dz \stackrel{\text{Residue thm.}}{=} 2\pi i \sum_{\text{poles in upper half plane}} \text{Res } f(z)$$

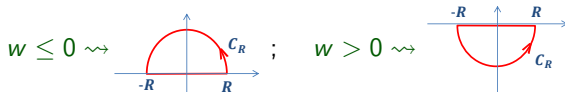
(iv) $|\int_{S_R} f(z) dz| \leq \max_{z \in S_R} |f(z)| \cdot L \leq \max_{|z|=R} |f(z)| \cdot \pi R \xrightarrow[R \rightarrow \infty]{\text{Must show } 0} 0$

Summary Lecture 25: Real integrals

- 3 Type III: $I = \int_{-\infty}^{\infty} f(x)e^{-iwx} dx$ f rational, no real singularities, ...

Same procedure as for Type II integrals...

...and choose C_R to insure $w \cdot \operatorname{Im} z \leq 0$:



$$\implies |f(z)e^{-i\omega z}| \stackrel{z=x+iy}{=} |f(z)|e^{\omega \cdot y} \stackrel{z \in C_R}{\leq} |f(z)|$$

$$\text{Re } I = \int_{-\infty}^{\infty} f(x) \cos(wx) dx \quad \text{Im } I = - \int_{-\infty}^{\infty} f(x) \sin(wx) dx$$