

Summary: Real integrals

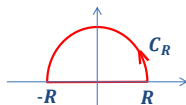
❶ Type I: $I = \int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta$ F rational

Substitution: $z = e^{i\theta}$, $\cos \theta = \frac{1}{2}(z + \frac{1}{z})$, $\dots$, $[0, 2\pi) \rightsquigarrow |z| = 1$

❷ Type II: $I = \int_{-\infty}^{\infty} f(x) dx$ f rational, no real singularities, $\dots$

(i) $I = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$

(ii) $\int_{-R}^R f(x) dx = \oint_{C_R} f(z) dz - \int_{S_R} f(z) dz$



(iii) R big enough $\Rightarrow C_R$ encircles all poles in upper half plane

$$\oint_{C_R} f(z) dz \stackrel{\text{Residue thm.}}{=} 2\pi i \sum_{\text{poles in upper half plane}} \text{Res } f(z)$$

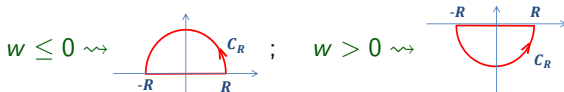
(iv) $|\int_{S_R} f(z) dz| \leq \max_{z \in S_R} |f(z)| \cdot L \leq \max_{|z|=R} |f(z)| \cdot \pi R \xrightarrow[R \rightarrow \infty]{\text{Must show } 0} 0$

Summary Lecture 25: Real integrals

- 3 Type III: $I = \int_{-\infty}^{\infty} f(x)e^{-iwx} dx$ f rational, no real singularities, ...

Same procedure as for Type II integrals...

...and choose C_R to insure $w \cdot \operatorname{Im} z \leq 0$:



$$\implies |f(z)e^{-i\omega z}| \stackrel{z=x+iy}{=} |f(z)|e^{\omega \cdot y} \stackrel{z \in C_R}{\leq} |f(z)|$$

$$\text{Re } I = \int_{-\infty}^{\infty} f(x) \cos(wx) dx \quad \text{Im } I = - \int_{-\infty}^{\infty} f(x) \sin(wx) dx$$

Lecture 26: Real integrals and singularities

Kreyszig: Section 16.4

- 1 Computing real integrals:

Type IV: $\int_{-\infty}^{\infty} f(x)dx$, f has order 1 poles on the real axis

- 2 Examples

Frist øving 12 og 13:

I dag kl 12:00!

Next week:

Exam problems, repetition. No lecture on Friday.

Lecture 26: Singular integrals, principal values

$f(x)$ singular at $x_0 \in (a, b)$:

$$\int_a^b f(x) dx := \left(\lim_{r \rightarrow 0^+} \int_a^{x_0-r} + \lim_{r \rightarrow 0^+} \int_{x_0+r}^b \right) f(x) dx$$

$$\text{pr.v. } \int_a^b f(x) dx := \lim_{r \rightarrow 0} \left(\int_a^{x_0-r} + \int_{x_0+r}^b \right) f(x) dx$$

Integral exists $\Rightarrow$ principal value exists
 $\nLeftarrow$

Lecture 26: Type IV real integrals

$I = \text{pr.v.} \int_{-\infty}^{\infty} f(x) dx$, f rational,
with order 1 poles on $\mathbb{R}$ at $a_1, \dots, a_m$

$$(i) \quad I = \lim_{\substack{R \rightarrow \infty \\ r \rightarrow 0}} I_{r,R}, \quad I_{r,R} := \left(\int_{-R}^{a_1-r} + \sum_{j=1}^{m-1} \int_{a_j+r}^{a_{j+1}-r} + \int_{a_m+r}^R \right) f(x) dx$$

$$(ii) \quad I_{r,R} = \oint_{C_{r,R}} f(z) dz - \left(\int_{S_R} + \sum_{j=1}^m \int_{S_{j,r}} \right) f(z) dz$$

$$(iii) \quad \oint_{C_{r,R}} f(z) dz \stackrel{\text{Residue thm.}}{=} 2\pi i \sum_{\substack{\text{poles in upper} \\ \text{half plane}}} \text{Res } f(z) \quad (R \text{ big, } r \text{ small})$$

$$(iv) \quad \left| \int_{S_R} f(z) dz \right| \leq \max_{|z|=R} |f(z)| \cdot \pi R \xrightarrow[R \rightarrow \infty]{\text{Must show}} 0$$

$$(v) \quad \text{Lemma: } \int_{S_{j,r}} f(z) dz \xrightarrow[r \rightarrow 0]{} \pi i \text{Res}_{z=a_j} f(z)$$

Summary Lecture 26: Real integrals

1 Principal value:

$f(x)$ singular at $x_0 \in (a, b)$

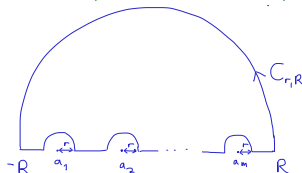
$$\Rightarrow \text{pr.v.} \int_a^b f(x) dx := \lim_{r \rightarrow 0} \left(\int_a^{x_0-r} + \int_{x_0+r}^b \right) f(x) dx$$

Integral exists $\Rightarrow$ principal value exists $\nRightarrow$ integral exists

2 Type IV: $I = \int_{-\infty}^{\infty} f(x) dx$ f rational, order 1 poles $a_1, \dots, a_m \in \mathbb{R}$

$$(i) I = \lim_{\substack{R \rightarrow \infty \\ r \rightarrow 0}} I_{r,R}, \quad I_{r,R} := \left(\int_{-R}^{a_1-r} + \sum_{j=1}^{m-1} \int_{a_j+r}^{a_{j+1}-r} + \int_{a_m+r}^R \right) f(x) dx$$

$$(ii) I_{r,R} = \oint_{C_{r,R}} f(z) dz - \left(\int_{S_R} + \sum_{j=1}^m \int_{S_{j,r}} \right) f(z) dz$$



Summary Lecture 26: Real integrals

(iii) R big and r small enough

$\Rightarrow C_{r,R}$ encircles all poles in upper half plane

$$\Rightarrow \oint_{C_{r,R}} f(z) dz \stackrel{\text{Residue thm.}}{=} 2\pi i \sum_{\substack{\text{poles in upper} \\ \text{half plane}}} \text{Res } f(z)$$

$$(iv) \left| \int_{S_R} f(z) dz \right| \leq \max_{z \in S_R} |f(z)| \cdot L \leq \max_{|z|=R} |f(z)| \cdot \pi R \xrightarrow[R \rightarrow \infty]{\text{Must show}} 0$$

$$(v) \text{ Lemma: } \int_{S_{j,r}} f(z) dz \xrightarrow[r \rightarrow 0]{} \pi i \text{Res}_{z=a_j} f(z)$$

