Laplace Transform

1. $F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}dt$

2. Unit step function: $u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$ $\boxed{\mathcal{L}[u(t-a)] = \frac{1}{s}e^{-as}}$

3. Delta-function: $\int_0^\infty \delta(t-a)f(t)\,dt = f(a)$ $\boxed{\mathcal{L}[\delta(t-a)] = e^{-as}}$

4. Properties:

Linearity: $\mathcal{L}[af(t) + bg(t)](s) = a\mathcal{L}[f](s) + b\mathcal{L}[g](s)$

s-shift: $\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a)$ for $s - a > k$

t-shift: $\boxed{\mathcal{L}[f(t-a)u(t-a)](s) = e^{-as}\mathcal{L}[f](s)}$

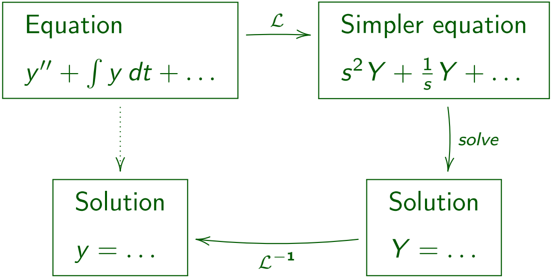
Derivatives: $\mathcal{L}[f'(t)](s) = s\mathcal{L}[f](s) - f(0)$

$$\mathcal{L}[f''(t)](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0)$$

Integral: $\mathcal{L}[\int_0^t f(\tau)d\tau](s) = \frac{1}{s}\mathcal{L}[f](s)$

Laplace Transform

5. Solving equations with the Laplace transform:



Lecture 3: Laplace Transform

Kreyszig: Sections 6.5 – 6.6, 11.1

1. On finding partial fractions
2. Convolutions
3. Integral representation of solutions of ODEs
4. Differentiation and integration of transforms
5. Examples

Partial fraction decomposition for $\frac{P(s)}{Q(s)}$

$P(s)$ and $Q(s)$ polynomials, no common factor, $\mathrm{order}(P) < \mathrm{order}(Q)$

1. $Q(s) = (s - s_1)(s - s_2)(s - s_3)\dots$ **non-repeated factors**

   $$\Rightarrow \qquad \frac{P(s)}{Q(s)} = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \frac{A_3}{s - s_3} + \dots$$

2. $Q(s) = (s - s_0)^n \dots$ **repeated factors**

   $$\Rightarrow \qquad \frac{P(s)}{Q(s)} = \frac{A_1}{s - s_0} + \frac{A_2}{(s - s_0)^2} + \dots + \frac{A_n}{(s - s_0)^n} + \dots$$

3. $Q(s) = (s^2 + b_1 s + a_1)\dots$ **irreducible, non-repeated quad. factors**

   $$\Rightarrow \qquad \frac{P(s)}{Q(s)} = \frac{A_1 s + B_1}{s^2 + b_1 s + a_1} + \dots$$

4. $Q(s) = (s^2 + b_1 s + a_1)^n \dots$ see earlier mathematics courses.

Partial fraction decomposition

Eksample 1:

$$\frac{s^5+2}{s(s+1)(s-1)^2(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{Es+F}{s^2+1}$$

Eksample 2:

$$\frac{1}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

Multiply by denominator $(s^2+1)(s^2+2s+2)$

$$\begin{aligned} 1 &= 1 + 0s + 0s^2 + 0s^3 \\ &= (As+B)(s^2+2s+2) + (Cs+D)(s^2+1) \\ &= (2B+D) + (2B+2A+C)s + (2A+B+D)s^2 + (A+C)s^3 \end{aligned}$$

Coefficients of same powers of s must coincide:

$$\begin{cases} 1 = 2B + D \\ 0 = 2B + 2A + C \\ 0 = 2A + B + D \\ 0 = A + C \end{cases} \Rightarrow \begin{cases} A = -\frac{2}{5} \\ B = \frac{1}{5} \\ C = \frac{2}{5} \\ D = \frac{3}{5} \end{cases}$$

Lecture 3: Laplace Transform

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}dt$$

Convolution

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

$$\mathcal{L}[\, f * g\,](s) = \mathcal{L}[f](s) \cdot \mathcal{L}[g](s)$$

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}dt$$

Derivatives and integrals of transforms

$$-F'(s) = \mathcal{L}[tf(t)](s)$$

$$\int_s^\infty F(\tau)\,d\tau = \mathcal{L}\left[\frac{1}{t}f(t)\right](s)$$

$f(x)$ is p-periodic if $f(x+p) = f(x)$ for all $x \in \mathbb{R}$

Representation of $f(x)$ by trigonometric series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big)$$

Summary Lecture 3: Laplace Transform

1. $F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}dt$

2. Properties:

   Linearity: $\mathcal{L}[af(t) + bg(t)](s) = a\mathcal{L}[f](s) + b\mathcal{L}[g](s)$

   s-shift: $\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a)$ for $s-a>k$

   t-shift: $\mathcal{L}[f(t-a)u(t-a)](s) = e^{-as}\mathcal{L}[f](s)$

   Derivatives: $\mathcal{L}[f'(t)](s) = s\mathcal{L}[f](s) - f(0)$

   $$\boxed{\mathcal{L}^{-1}[F'(s)](t) = -tf(t)}$$

   Integral: $\mathcal{L}[\int_0^t f(\tau)d\tau](s) = \frac{1}{s}\mathcal{L}[f](s)$

   $$\boxed{\mathcal{L}^{-1}[\int_s^\infty F(\bar{s})d\bar{s}](t) = \frac{1}{t}f(t)}$$

   Convolution: $\boxed{\mathcal{L}[f*g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)}$ $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

3. Unit step, delta functions: $\mathcal{L}[u(t-a)] = \frac{1}{s}e^{-as}$, $\mathcal{L}[\delta(t-a)] = e^{-as}$