Fourier series

1. Periodic functions:
   p-periodic: $f(x) = f(x+p) \;\; (= f(x+np),\; n \in \mathbb{N}), \quad x \in \mathbb{R}$.

   2π-periodic: $a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big)$

2. Trigonometric system:
   $\mathcal{T} = \{1, \cos x, \sin x, \ldots, \cos nx, \sin nx, \ldots\}$ (2π-periodic)

   Orthogonality:

   $$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x) g(x) dx = 0 \quad \text{for} \quad f, g \in \mathcal{T},\; f \neq g.$$

3. Fourier series for 2π-periodic $f(x)$:

   $S_f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big)$ where

   $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,$$

   $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx,$$

   $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx$$

Fourier series

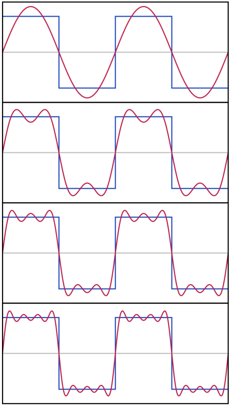

Figure: From Wikipedia

5. Convergence and sum

Assume

(A1) f periodic and piecewise continuous,

(A2) f has both right and left derivatives at x.

Then the Fourier series S_f converge at x and

$$S_f(x) = \frac{f(x^+) + f(x^-)}{2},$$

i.e. $S_f(x) = f(x)$ if f is also continuous at x.

`Animation by Yuya Suzuki` [a]

[a] Oscillations near discontinuities do not die out – Gibb's phenomenon!

Lecture 5: Fourier Series

Kreyszig: Section K9_11.4, 11.2, Folland p 38-39

1. Complex Fourier series
2. Fourier series with periode $2L$
3. Linearity, (term-wise) derivation and integration
4. Examples

Section 11.4 in 9th ed. available on course wiki page (Fremdriftsplan).

Folland is available on Blackboard.

Homework: Read yourselves Kreyszig section 11.3.

Complex Fourier series of 2π-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$= \boxed{\sum_{n=-\infty}^{\infty} c_n e^{inx}}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$\boxed{c_{\pm n} = \tfrac{1}{2}(a_n \mp i b_n), \quad c_0 = a_0}$$

Fourier series of $p = 2L$-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx,$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi x}{L}}\,dx$$

Lecture 5: Fourier Analysis

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Linearity:

$$S_{K_1 f_1 + K_2 f_2}(x) = K_1 S_{f_1}(x) + K_2 S_{f_2}(x)$$

(Term-wise) derivation:

$$S_{f'}(x) = \sum_{n=-\infty}^{\infty} (inc_n)\, e^{inx}$$

$$= \sum_{n=1}^{\infty} \left((nb_n) \cos nx - (na_n) \sin nx\right)$$

Summary Lecture 5: Fourier series

1. Complex Fourier series of $p = 2\pi$-periodic $f(x)$:

$$S_f(x) = \boxed{\sum_{n=-\infty}^{\infty} c_n e^{inx}} = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$\boxed{c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx}, \qquad \boxed{c_{\pm n} = \frac{1}{2}(a_n \mp i b_n)}$$

2. Fourier series of $p = 2L$-periodic $f(x)$:

$$\boxed{S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right)} = \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi x}{L}}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx,$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}}\, dx$$

Summary Lecture 5: Fourier series

4. Linearity:

$$S_{K_1 f_1 + K_2 f_2}(x) = K_1 S_{f_1}(x) + K_2 S_{f_2}(x)$$

(Term-wise) derivation: $p = 2\pi$

$$S_{f'}(x) = \sum_{n=-\infty}^{\infty} (inc_n)\, e^{inx} = \sum_{n=1}^{\infty} \big((nb_n)\cos nx - (na_n)\sin nx\big)$$

(Term-wise) integration: when $\int_{-\pi}^{\pi} f(x)\,dx = 0$ $(p = 2\pi)$

$$S_{\int_0^x f\,dx}(x) = C_0 + \sum_{n=-\infty}^{\infty} \frac{c_n}{in}\, e^{inx}$$

$$= A_0 + \sum_{n=1}^{\infty} \big(-\tfrac{b_n}{n}\cos nx + \tfrac{a_n}{n}\sin nx\big)$$

where $C_0 = A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\int_0^s f\ ds)\,dx$