# Fourier series

1. Fourier series of $p = 2\pi$-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx}, \qquad \boxed{c_n = \frac{1}{2}(a_n - ib_n)}$$

2. Fourier series of $p = 2L$-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x) dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx,$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}}\, dx$$

# Fourier series

4. Linearity:

$$S_{K_1 f_1 + K_2 f_2}(x) = K_1 S_{f_1}(x) + K_2 S_{f_2}(x)$$

(Term-wise) derivation: $p = 2\pi$

$$S_{f'}(x) = \sum_{n=-\infty}^{\infty} (inc_n)\, e^{inx} \qquad [\, S_f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx} \,]$$

(Term-wise) integration: when $\int_{-\pi}^{\pi} f(x)\, dx = 0$ $(p = 2\pi)$

$$S_{\int_0^x f\, dx}(x) = C_0 + \sum_{n=-\infty}^{\infty} \frac{c_n}{in}\, e^{inx} \quad \text{where} \quad C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\int_0^x f\, ds\right) dx$$

# Lecture 6: Fourier Series

Kreyszig: Section 11.2, 11.4

1. Odd and even functions
2. Fourier sin and cos series
3. Odd and even periodic extensions
4. Approximation by trigonometric polynomials
5. Examples

**Homework:** Read yourselves Kreyszig section 11.3.

## Odd and even functions

Even $g(-x) = g(x)$

$$\boxed{\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx}$$

Odd $h(-x) = -h(x)$

$$\boxed{\int_{-L}^{L} h(x)dx = 0}$$

even $\cdot$ even $=$ odd $\cdot$ odd $=$ even; odd $\cdot$ even $=$ odd

Fourier cos series:

$f$ even: $S_f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$,

$$a_0 = \frac{1}{L} \int_0^L f(x) dx, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

Fourier sin series:

$f$ odd: $S_f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Even $2L$-periodic extensions of $f(x)$, $0 \leq x \leq L$

$f_1(x)$, $x \in \mathbb{R}$: $f_1 = f$ on $[0, L]$, even, $2L$-periodic

$S_{f_1}(x) =$ cos-series $=:$ the Fourier cos series of $f$

Odd $2L$-periodic extensions of $f(x)$, $0 \leq x \leq L$

$f_2(x)$, $x \in \mathbb{R}$: $f_2 = f$ on $[0, L]$, odd, $2L$-periodic

$S_{f_2}(x) =$ sin-series $=:$ the Fourier sin series of $f$

## Approximation of $f$ by trigonometric polynomial

$$P_k(x) = A_0 + \sum_{n=1}^{k} (A_n \cos nx + B_n \sin nx)$$

---

$$S_{f,k}(x) = a_0 + \sum_{n=1}^{k} (a_n \cos nx + b_n \sin nx),$$

is the best means square approximation (least error):

$$\|f - S_{f,k}\|^2 \leq \|f - P_k\|^2 \quad \text{for all } P_k(x)$$

---

where $\quad \|g\|^2 = \int_{-\pi}^{\pi} |g(x)|^2 dx$

# Summary Lecture 6: Fourier series

1. Odd and even functions

   Even $g(-x) = g(x)$: $\int_{-L}^{L} g(x)dx = 2\int_0^L g(x)dx$

   Odd $h(-x) = -h(x)$: $\int_{-L}^{L} h(x)dx = 0$

   even · even = odd · odd = even; odd · even = odd

2. Fourier sin and cos series:

   $f$ even: $S_f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$,

   $$\boxed{a_0 = \frac{1}{L}\int_0^L f(x)dx}, \quad \boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos \frac{n\pi x}{L} dx}$$

   $f$ odd: $S_f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$, $\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin \frac{n\pi x}{L} dx}$

3. Even and odd $2L$-periodic extensions:

   $f(x)$, $0 \le x \le L$

   $f_1(x)$, $x \in \mathbb{R}$: $f_1 = f$ on $[0, L]$, even, $2L$-periodic

   $S_{f_1}(x) =$ cos-series $=:$ the Fourier cos series of $f$

   $f_2(x)$, $x \in \mathbb{R}$: $f_2 = f$ on $[0, L]$, odd, $2L$-periodic

   $S_{f_2}(x) =$ sin-series $=:$ the Fourier sin series of $f$

# Summary Lecture 6: Fourier series

4. Fourier series of $2\pi$-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$S_{f,k}(x) = a_0 + \sum_{n=1}^{k}(a_n \cos nx + b_n \sin nx) \qquad k\text{-th partial sum}$$

5. Approximation of $f(x)$ by trigonometric polynomial

$$P_k(x) = A_0 + \sum_{n=1}^{k}(A_n \cos nx + B_n \sin nx)$$

$S_{f,k}(x)$ best mean square (or $L^2$) approximation (least error):

$$\boxed{\|f - S_{f,k}\|^2 \leq \|f - P_k\|^2} \quad \text{for all } k\text{-order trig. poly'als } P_k(x),$$

where $\quad \|g\|^2 := \int_{-\pi}^{\pi} |g(x)|^2 dx$

Obs: $\|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{k}(a_n^2 + b_n^2)\right]$