Fourier series

1. Fourier series of $p = 2L$-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}}$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \quad b_n = \ldots \sin \ldots,$$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{n\pi x}{L}}\, dx$$

2. Fourier sin and cos series:

f even: $S_f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$, $a_0 = \frac{1}{L} \int_0^L \ldots, a_n = \frac{2}{L} \int_0^L \ldots$

f odd: $S_f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$, $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

3. Even and odd $2L$-periodic extensions of $f(x)$, $x \in [0, L]$:

$f_1(x)$, $x \in \mathbb{R}$: $f_1 = f$ on $[0, L]$, even, $2L$-periodic

$S_{f_1}(x) =$ the Fourier cos series of f

$f_2(x)$, $x \in \mathbb{R}$: $f_2 = f$ on $[0, L]$, odd, $2L$-periodic

$S_{f_2}(x) =$ the Fourier sin series of f

Fourier series

4. Linearity, derivation and integration: $(p = 2\pi)$

$$S_{K_1 f_1 + K_2 f_2}(x) = K_1 S_{f_1}(x) + K_2 S_{f_2}(x)$$

$$S_{f'}(x) = \sum_{n=-\infty}^{\infty} (inc_n)\, e^{inx} \qquad [\, S_f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx} \,]$$

$$S_{\int_0^x f\, dx}(x) \underset{\int_{-\pi}^{\pi} f\, dx = 0}{=} C_0 + \sum_{n=-\infty}^{\infty} \frac{c_n}{in}\, e^{inx}, \qquad C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_0^x f\, ds\right)dx$$

5. Approximation of $f(x)$ by trigonometric polynomial: $(p = 2\pi)$

$$P_k(x) = A_0 + \sum_{n=1}^{k}(A_n \cos nx + B_n \sin nx), \qquad S_{f,k} \ (\underset{k\to\infty}{\to} S_f)$$

$S_{f,k}(x)$ best mean square (L^2) approximation (least error):

$$\boxed{\|f - S_{f,k}\|^2 \leq \|f - P_k\|^2} \quad \text{for all } P_k(x), \qquad \|g\|^2 := \int_{-\pi}^{\pi} |g|^2 dx$$

Obs: $\|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{k}(a_n^2 + b_n^2)\right]$

Lecture 7: Fourier Series

Kreyszig: Section 11.4, Folland: Section 2.3, Convergence proof note

1. Bessel's inequality, Parseval's identity
2. Proof of point-wise convergence
3. Uniform convergence and regularity/decay
4. Examples

The convergence proof note is available on wiki

Copy of Folland is available on Blackboard

$$\|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{k} (a_n^2 + b_n^2) \right]$$

Bessel's inequality

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = 2 \sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx$$

Parseval's identity

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = 2 \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx$$

$$S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \overset{def}{=} \lim_{N\to\infty} \underbrace{\sum_{n=-N}^{N} c_n e^{inx}}_{S_{f,N}(x)}$$

Pointwise convergence of Fourier series:

$f(x)$ is 2π-periodic, piecewise continuous, $f'(a)$ exists

$\Longrightarrow \quad S_f(a)$ converges and $S_f(a) = f(a)$.

Pointwise convergence of Fourier series

Assume: $f(x)$ is a real piecewise continuous 2π-periodic function

(f piecewise continuous $\implies \int_{-\pi}^{\pi} |f(x)|^2 dx \le 2\pi \max_{x\in[-\pi,\pi]} |f(x)|^2 < \infty$)

Fourier series of f: $$S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

Partial sums: $$S_{f,N}(x) = \sum_{n=-N}^{N} c_n e^{inx}, \qquad S_f(x) := \lim_{N\to\infty} S_{f,N}(x)$$

Bessel's inequality: $$\sum_{n=-\infty}^{\infty} |c_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$$

Riemann-Lebesgue's lemma: $\lim_{n\to\pm\infty} c_n = 0$

Theorem (Convergence and sum)

If f is piecewise continuous and $f'(a)$ exists, then $S_f(a) = f(a)$.

Proof

From P.R. Chernoff: Pointwise convergence of Fourier Series. *Amer. Math. Monthly* 87(5): 399–400, 1980

Assume first $a = 0$ and $f(a) = f(0) = 0$:

1. Let $\tilde{f}(x) = \dfrac{f(x)}{e^{ix} - 1}$, $x \neq 0$; $\quad \tilde{f}(0) = \frac{1}{i} f'(0)$:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{ix} - 1) \tilde{f}(x) e^{-inx} dx = \tilde{c}_{n-1} - \tilde{c}_n$$

$$\implies \quad S_{f,N}(0) = \sum_{n=-N}^{N} c_n e^{in \cdot 0} = \sum_{n=-N}^{N} c_n = \sum_{n=-N}^{N} (\tilde{c}_{n-1} - \tilde{c}_n) = \tilde{c}_{-N-1} - \tilde{c}_N$$

2. $\tilde{f}(x)$ is piecewise continuous $\qquad \left(\tilde{f}(x) \underset{|x| \ll 1}{\approx} \frac{f(x)}{ix} \underset{x \to 0}{\to} \frac{1}{i} f'(0) = \tilde{f}(0), \;\; x \neq 0 \text{ ok} \right)$

$$\implies \lim_{N \to \pm\infty} \tilde{c}_N = 0 \text{ by Riemann-Lebesgue} \quad \implies \quad \lim_{N \to \infty} S_{f,N}(0) = 0 = f(0)$$

For general $a, f(a)$: Let $g(x) = f(x + a) - f(a)$.

Then $g(0) = 0$, $g'(0) = f'(a)$, and $S_{g,N}(x) = S_{f,N}(x + a) - f(a)$. Hence

2. $\quad \implies \quad |S_{f,N}(a) - f(a)| = |S_{g,N}(0)| \to 0$ as $N \to \infty$.

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$$S_f(x) \overset{def}{=} \lim_{N\to\infty} S_{f,N}(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{inx}$$

Uniform convergence of Fourier series:

$f(x)$ is 2π-periodic, continuous, f' piecewise cont.

$\implies S_f(x) = f(x)$ converges absolutely and uniformly

$$S_f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{inx}$$

$$S_{f'}(x) = \sum_{n=-\infty}^{\infty} (in) c_n \, e^{inx}$$

Decay of coefficients and derivaties:

(a) $f^{(k)}$ piecewise cont. $\implies$ $n^k a_n, \; n^k b_n, \; n^k c_n \to 0,$

(b) $|c_n| \leq C|n|^{-k-\alpha}, \; \alpha > 1 \quad (\iff |a_n| + |b_n| \leq C|n|^{-k-\alpha})$

$\implies \quad f^{(k)} = S_{f^{(k)}}$ exists and is continuous

Summary Lecture 7: Fourier series

1. Fourier series of 2π-periodic $f(x)$:

   $S_f(x) \;= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

2. Parseval's identity (f p.w. cont.)

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = 2\sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx$$

   Bessel's inquality: $\cdots \leq \ldots$

3. Pointwise convergence:

   (a) (Today, with proof) $S_f(a)$ converges and $= f(a)$,
   if f 2π-periodic, p.w. continuous, and $f'(a)$ exists.

   (b) (Previously, no proof) $S_f(a)$ converges, $= \frac{1}{2}(f(a^-) + f(a^+))$,
   if f 2π-periodic, p.w. continuous, and $\frac{d^\pm}{dx}(a)$ exists.

4. Uniform convergence: S_f converges uniformly and absolutely to f,
   if f 2π-periodic, continuous, and f' p.w. continuous.

Summary: Fourier series

5. Decay and differentialbility: f 2π-periodic.

   (a) $f^{(k)}$ p.w. continuous, $f^{(k-1)}$ continuous

   $$\implies \quad n^k a_n,\ n^k b_n,\ n^k c_n \to 0$$

   (b) $|c_n| \leq C|n|^{-k-\alpha}$, $\alpha > 1$

   $$\left(\iff |a_n| + |b_n| \leq C|n|^{-k-\alpha},\ \alpha > 1 \right)$$

   $$\implies \quad f^{(k)} = S_{f^{(k)}} \text{ exists and is continuous}$$