Summary Lecture 8: Fourier series

1. Fourier series of 2π-periodic $f(x)$:

   $$S_f(x) \;= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

2. Parseval's identity   (f p.w. cont.)

   $$\boxed{2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = 2\sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx}$$

   Bessel's inquality: $\cdots \leq \ldots$

3. Pointwise convergence: $S_f(a)$ converges to $\frac{1}{2}(f(a^-) + f(a^+))$ ($f(a)$),

   if f 2π-periodic, p.w. continuous, and $\frac{d^{\pm} f}{dx}(a)$ exists ($f'(a)$ exists).

4. Uniform convergence: S_f converges uniformly and absolutely to f,

   if f 2π-periodic, continuous, and f' p.w. continuous.

Summary: Fourier series

5. Linearity, derivation: $(p = 2\pi)$

$$S_{K_1 f_1 + K_2 f_2}(x) = K_1 S_{f_1}(x) + K_2 S_{f_2}(x)$$

$$S_{f'}(x) = \sum_{n=-\infty}^{\infty} (inc_n)\, e^{inx} \qquad \left[\, S_f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx} \,\right]$$

6. Decay and differentialbility:

(a) $f^{(k)}$ p.w. cont., $f^{(k-1)}$ cont. $\implies$ $n^k a_n,\ n^k b_n,\ n^k c_n \to 0$

(b) $|a_n| + |b_n| = 2|c_n| \le \frac{C}{|n|^{k+\alpha}},\ \alpha > 1 \implies f^{(k)} = S_{f^{(k)}}$ exists, is cont.

7. Fourier series of $2L$-periodic f:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}},$$

$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}}\, dx}$$

Lecture 8: Fourier integrals and transforms

Kreyszig: Section 11.7, 11.9

1. Fourier integral
2. Fourier transform
3. Properties
4. Examples

Homework: Repeat complex *numbers*, *absolute values*, *exponentials* [Mat 3]

Fourier integral of $f(x)$, $x \in \mathbb{R}$:

$$I_f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw,$$

where

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx.$$

Lecture 8: Fourier transform

Fourier transform of $f(x)$, $x \in \mathbb{R}$:

$$\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Invers:

$$\mathcal{F}^{-1}[g](x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{iwx} dw$$

Properties:

$$\mathcal{F}[\,af(x) + bg(x)\,](w) = a\,\mathcal{F}[f](w) + b\,\mathcal{F}[f](w)$$

$$\mathcal{F}[\,f'\,](w) = \,(iw)\mathcal{F}[f](w) \qquad\qquad (|f(x)| \underset{|x|\to\infty}{\longrightarrow} 0)$$

$$\mathcal{F}[\,e^{-iax}f(x)\,](w) = \mathcal{F}[f](w+a)$$

Summary: Fourier integral and transform

1. Fourier integral of $f(x)$, $x \in \mathbb{R}$:

$$I_f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\, dw \qquad \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$$

2. Fourier transform of $f(x)$, $x \in \mathbb{R}$:

$$\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$$

Invers: $$\mathcal{F}^{-1}[g](x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{iwx}\, dw$$

Under certain conditions: $f(x) = I_f(x) = \mathcal{F}^{-1}[\mathcal{F}[f]](x)$

3. Properties:

$$\mathcal{F}[\, af(x) + bg(x)\,](w) = a\,\mathcal{F}[f](w) + b\,\mathcal{F}[f](w)$$

$$\mathcal{F}[\, f'\,](w) = (iw)\mathcal{F}[f](w) \qquad (|f(x)| \to 0 \text{ as } |x| \to \infty)$$

$$\mathcal{F}[\, e^{-iax} f(x)\,](w) = \mathcal{F}[f](w + a)$$