Summary: Fourier integral and transform

1. Fourier integral of $f(x)$, $x \in \mathbb{R}$:

$$I_f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw \qquad \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

2. Fourier transform of $f(x)$, $x \in \mathbb{R}$:

$$\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Invers: $$\mathcal{F}^{-1}[g](x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{iwx} dw$$

Under certain conditions: $f(x) = I_f(x) = \mathcal{F}^{-1}[\mathcal{F}[f]](x)$

3. Properties:

$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$	$\mathcal{F}^{-1}[a\hat{f} + b\hat{g}] = af + bg$
$\mathcal{F}[f'](w) = (iw)\mathcal{F}[f](w)$	$\mathcal{F}^{-1}[\hat{f}'(w)](x) = (-ix)f(x)$
$\mathcal{F}[e^{-iax} f(x)](w) = \mathcal{F}[f](w + a)$	$\mathcal{F}^{-1}[e^{iaw} \hat{f}(w)](x) = f(x + a)$

Lecture 9: Fourier transform and partial differential equations

Kreyszig: Section 11.9, 12.1

1. Fourier transform
2. Convolution
3. Introduction to partial differential equations
4. Examples

Homework:

1. Read Kreyszig 12.2 yourselves.
2. Repeat *Solution of 1st and 2nd order differential equations* [Mat 3]

Important transform:

$$\mathcal{F}[e^{-ax^2}](w) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$$

Remember: $\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$

$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$	$\mathcal{F}^{-1}[a\hat{f} + b\hat{g}] = af + bg$
$\mathcal{F}[f'](w) = (iw)\mathcal{F}[f](w)$	$\mathcal{F}^{-1}[\hat{f}'(w)](x) = (-ix)f(x)$
$\mathcal{F}[e^{-iax} f(x)](w) = \mathcal{F}[f](w + a)$	$\mathcal{F}^{-1}[e^{iaw} \hat{f}(w)](x) = f(x + a)$

Convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

$$\boxed{\mathcal{F}[f * g](w) = \sqrt{2\pi}\,\mathcal{F}[f](w) \cdot \mathcal{F}[g](w)}$$

Remember:

$$\mathcal{F}[f](w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx$$

Partial differential equations (PDEs)

Example: $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (\text{or} \quad u_t = c^2 u_{xx})$$

Order, linear, homogeneous, solution

Linear + homogeneous $\Rightarrow$ Superposition

Uniqueness: Need boundary and initial conditions

2nd order PDEs

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

Type	Condition	Typical example
Hyperbolic	$AC - B^2 < 0$	$u_{tt} - u_{xx} = 0$ (wave eqn)
Parabolice	$AC - B^2 = 0$	$u_t - u_{xx} = 0$ (heat eqn)
Elliptic	$AC - B^2 > 0$	$u_{xx} + u_{yy} = 0$ (Laplace eqn)

Summary Lecuter 9: Fourier transform and PDEs

1. Fourier transform $\mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$

   $$\boxed{\mathcal{F}[e^{-ax^2}](w) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}}$$

   $$\boxed{\mathcal{F}[f * g](w) = \sqrt{2\pi}\, \mathcal{F}[f](w) \cdot \mathcal{F}[g](w)} \qquad f * g(x) = \int_{-\infty}^{\infty} f(y) g(x-y) dy$$

2. Partial differential equations (PDEs)

   Equations involving partial derivatives of the unknown

   Concepts: Order, linear, homogeneous, hyperbolic/parabolic/elliptic

   Solution: u solution of PDE in region R if

   (i) all derivatives appearing in PDE exist and are continuous in R

   (ii) u satisfy the PDE in all points in R

   Superposition/Linearity:

   u_1 and u_2 solve same *linear, homogeneous* PDE in R; $a, b \in \mathbb{R}$
   $\Rightarrow au_1 + bu_2$ solves same PDE in R

   Unique solution:

   Need also boundary and initial conditions!