NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4120 Mathematics 4K**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: (Code C): Approved simple calculator.

Other information:
Every answer must be justified; describe clearly how you have reached your answers.

The exam has 8 problems, 1, 2, 3-a, 3-b, 4, 5, 6-a, 6-b, all of which will be given equal weight when the grade is computed.

Language: English

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Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
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Date Signature

Problem 1 (i) Find the solution $y(t)$ of the Volterra integral equation

$$y(t) - 2\int_0^t e^{-2\tau} y(t-\tau)\,d\tau = e^{-t}, \qquad t \geq 0.$$

(ii) Find the inverse Laplace-transform of the function

$$F(s) = \frac{3s+3}{s^2+2s+2}.$$

Problem 2 Let $f(x) = \frac{\pi}{2}\sin x$ be defined for $x \in [0,\pi]$. The Fourier cosine series of $f(x)$ is given by

$$S(x) = 1 - 2\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos(2nx).$$

(i) Sketch $S(x)$ on the interval $[-2\pi, 2\pi]$, (ii) determine the sum S_0 of the series

$$S_0 = -\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}, \quad \text{and}$$

(iii) show that $S(x)$ is uniformly and absolutely convergent for $x \in [-\pi, \pi]$.

Hint: Weierstrass M-test. You can assume that $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges when $k > 1$.

Problem 3 The temperature in a rod with increasing heat conductivity over time is modeled by the partial differential equation and boundary conditions

(1) $$\left(\frac{1}{1+2t}\right)\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < \pi, \qquad t > 0,$$

(2) $$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \qquad t \geq 0.$$

a) Find all solutions on the form $u(x,t) = F(x)G(t)$ of (1) and (2).

b) We also have a condition at $t = 1$,

(3) $$u(x,1) = 10\cos x + \sum_{n=2}^{\infty} \frac{\cos(nx)}{n^2}, \qquad 0 \leq x \leq \pi.$$

Find a solution $u(x,t)$ of the problem (1), (2) and (3) for $t > 0$.

Problem 4 Let

$$f(z) = iz - \frac{1}{z} \qquad \text{and} \qquad g(z) = e^{-|z|^2}.$$

Show that $f(z)$ is analytic for $z \neq 0$, but that $g(z)$ is not analytic in any point.

Problem 5 The Laurent series

$$\frac{e^{\frac{1}{z}}}{(z^2+4)(z-1)} = \sum_{n=0}^{\infty} a_n (z-1)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-1)^n}$$

converges at the point $z = 3$. At which of the points

$$z_1 = -2, \qquad z_2 = 1 + 2i \qquad \text{and} \qquad z_3 = \tfrac{1}{2} - \tfrac{1}{2}i,$$

does it also converge? Justify your answer.

Problem 6

a) Let S_R be the semicircle $z = Re^{i\theta}$ for $\theta \in [0, \pi]$. Show that

$$\int_{S_R} \frac{e^{iz}}{(z^2+4)^2}\, dz \to 0 \qquad \text{as} \qquad R \to \infty.$$

b) Use the result from a) and the Residue theorem to determine the value of the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)^2}\, dx.$$

Miscellaneous

- **Heaviside function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero except at $t = a$, $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, and $\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = g(a)$ for any continuous function g.
- **Convolution**

  For functions defined on the real line:
  $$f * g\,(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}.$$
  For functions defined only on the positive half-axis:
  $$f * g\,(x) = \int_{0}^{x} f(y)g(x-y)dy, \quad x > 0.$$

Laplace transform

- Definition: $\mathcal{L}[f](s) = F(s) = \int_0^\infty f(t)e^{-st}dt$

General formulas		$f(t)$	$F(s)$
		1	$\frac{1}{s}$
$\mathcal{L}[e^{at}f(t)](s) = F(s-a)$		$t^n, n = 1, 2, ...$	$\frac{n!}{s^{n+1}}$
$\mathcal{L}[f'](s) = s\mathcal{L}[f] - f(0)$		e^{at}	$\frac{1}{s-a}$
$\mathcal{L}[f''](s) = s^2\mathcal{L}[f] - sf(0) - f'(0)$		$t^n e^{at}, n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}$
$\mathcal{L}\left[\int_0^t f(\tau)d\tau)\right](s) = \frac{1}{s}\mathcal{L}[f]$		$\cos bt$	$\frac{s}{s^2+b^2}$
$\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)$		$\sin bt$	$\frac{b}{s^2+b^2}$
$\mathcal{L}[f(t-c)u(t-c)](s) = e^{-cs}F(s),\ c > 0$		$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathcal{L}[tf(t)](s) = -F'(s)$		$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$\mathcal{L}\left[\frac{f(t)}{t}\right](s) = \int_s^\infty F(\sigma)d\sigma$		$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
		$\delta(t-c), c > 0$	e^{-cs}

Fourier series and Fourier transform

- $2L$-periodic functions, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}},$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx,$$

$$c_0 = a_0, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx, \qquad c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \bar{c}_n.$$

- Functions defined on the whole real line (need not be periodic)

$$\hat{f}(w) = \mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx,$$

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw.$$

- Parseval's identities

$$\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2, \qquad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw$$

General formulas		$f(x)$	$\hat{f}(w)$
$\widehat{f'(x)} = iw\hat{f}(w)$		$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}}e^{-iaw}$
$\widehat{f''(x)} = -w^2\hat{f}(w)$		$\begin{cases}1, & -b \le x \le b\\ 0, & \lvert x\rvert > b\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin bw}{w}$
$f\widehat{(x-a)} = e^{-iaw}\hat{f}(w)$		$e^{-ax}u(x)$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
$\hat{f}(w-b) = e^{\widehat{ibx}f(x)}$		$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert w\rvert}}{a}$
$\widehat{f*g} = \sqrt{2\pi}\hat{f}\hat{g}$		e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-w^2/(4a)}$

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i \sin y)$
- $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cosh z = \frac{e^z+e^{-z}}{2}$, $\sinh z = \frac{e^z-e^{-z}}{2}$
- Taylor and Laurent series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz,$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \qquad b_n = \frac{1}{2\pi i} \oint_C f(z)(z-z_0)^{n-1} dz$$

Some useful integrals

$\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$

$\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$

$\int x^2 \sin ax \, dx = \frac{2}{a^2} x \sin ax + \frac{2-a^2x^2}{a^3} \cos ax + C$

$\int x^2 \cos ax \, dx = \frac{2}{a^2} x \cos ax - \frac{2-a^2x^2}{a^3} \sin ax + C$

$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$

$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$

$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \ a > 0$

Some trigonometric identities

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

Some important series

- $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$, $\sum_{n=0}^{\infty} x^n$ diverges for $|x| \geq 1$.
- $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for $x \in \mathbb{R}$.

Linear second order differential equations

Let r_1 and r_2 solve $r^2 + ar + b = 0$. Then

$$y''(x) + ay'(x) + by = 0$$

has general solution given by:

$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ if $r_1 \neq r_2, \quad r_1, r_2 \in \mathbb{R}$,

$y(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x}$ if $r_1 = r_2, \quad r_1, r_2 \in \mathbb{R}$,

$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ if $r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta, \quad \alpha, \beta \in \mathbb{R}$.