Problem 1 (i) Find the solution $y(t)$ of the Volterra integral equation

$$y(t) - 2\int_0^t e^{-2\tau} y(t-\tau)\,\mathrm{d}\tau = e^{-t}, \quad t \geq 0.$$

(ii) Find the inverse Laplace transform of the function

$$F(s) = \frac{3s+3}{s^2+2s+2}.$$

Solution. (i) We use the Laplace transform and note the convolution

$$\int_0^t e^{-2\tau} y(t-\tau)\,\mathrm{d}\tau = e^{-2t} * y(t).$$

Laplace transforming and using the convolution theorem we get

$$\begin{aligned}
& \mathcal{L}\left[y(t) - 2\int_0^t e^{-2\tau} y(t-\tau)\,\mathrm{d}\tau\right](s) = \mathcal{L}\left[e^{-t}\right](s) \\
\Leftrightarrow\quad & \mathcal{L}\left[y(t)\right](s) - 2\mathcal{L}\left[e^{-2t} * y(t)\right](s) = \mathcal{L}\left[e^{-t}\right](s) \\
\Leftrightarrow\quad & Y(s) - 2\mathcal{L}\left[e^{-2t}\right](s) \cdot \mathcal{L}\left[y(t)\right](s) = \mathcal{L}\left[e^{-t}\right](s) \\
\Leftrightarrow\quad & Y(s) - 2\frac{1}{s+2}Y(s) = \frac{1}{s+1} \\
\Leftrightarrow\quad & Y(s)\left(1 - \frac{2}{s+2}\right) = \frac{1}{s+1} \\
\Leftrightarrow\quad & Y(s)\frac{s}{s+2} = \frac{1}{s+1} \\
\Leftrightarrow\quad & Y(s) = \frac{s+2}{s(s+1)}.
\end{aligned}$$

Taking the inverse Laplace transform, expanding in partial fractions and looking up the resulting fractions in the table, gives us

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right](t) = \mathcal{L}^{-1}\left[\frac{s+2}{s(s+1)}\right](t) = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{1}{s+1}\right](t) = 2 - e^{-t}.$$

Alternatively: Complete the square to get $Y(s) = \frac{s+2}{\left(s+\frac{1}{2}\right)^2 - \frac{1}{4}} = \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 - \frac{1}{4}} + 3\frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 - \frac{1}{4}}$ and then by the tables (the sinh and cosh formulas were not available on the exam!)

$$\mathcal{L}^{-1}\left[Y\right] = e^{-\frac{1}{2}t}\cosh\left(\frac{1}{2}t\right) + 3e^{-\frac{1}{2}t}\sinh\left(\frac{1}{2}t\right) = 2 - e^{-t}.$$

(ii) Simply using the table after completing the square we get

$$f(t) = \mathcal{L}^{-1}\left[\frac{3s+3}{s^2+2s+2}\right](t) = 3\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+1}\right](t) = 3e^{-t}\cos(t).$$

Alterntive: Complete the square and then use the first shift theorem,

$$f(t) = 3\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+1}\right](t) = 3e^{-t}\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right](t) = 3e^{-t}\cos(t).$$

Problem 2 Let $f(x) = \frac{\pi}{2}\sin(x)$ be defined for $x \in [0,\pi]$. The Fourier cosine series of f is given by

$$S(x) = 1 - 2\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2nx).$$

(i) Sketch $S(x)$ on the interval $[-2\pi, 2\pi]$, (ii) determine the value S_0 of the series

$$S_0 = -\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1},$$

and (iii) show that $S(x)$ converges uniformly and absolutely for $x \in [-\pi,\pi]$.

Hint: Weierstrass M-test. You can assume that $\sum_{n=1}^{\infty}\frac{1}{n^k}$ converges for $k > 1$.

Solution. (i) Since f is defined on $[0, L]$ with $L = \pi$, the Fourier cosine series S of f is the Fourier series of the even 2π-periodic extension of f. This extension $\tilde{f}$ is continuous since f is continuous and $f(0) = 0 = f(\pi)$. Since $f'(x) = \frac{\pi}{2}\cos(x)$ for $x \in (0,\pi)$, $\lim_{x\to 0^+} f'(x) = \frac{\pi}{2}$, and $\lim_{x\to\pi^-} f'(x) = -\frac{\pi}{2}$, the left- and right-hand derivatives of $\tilde{f}$ exists in all points. Thus by the pointwise convergence result for Fourier series, $S(x) = \tilde{f}(x)$ in every point. We plot $\tilde{f}$ in fig. 1.

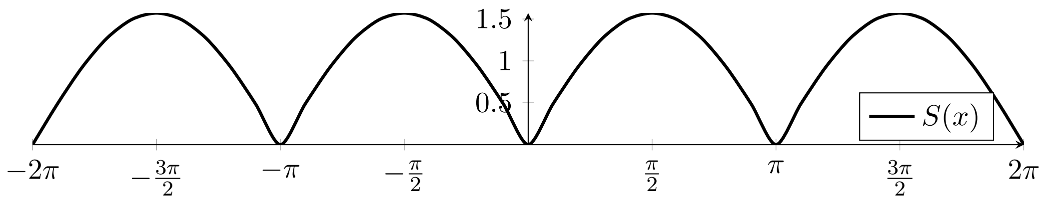


Figure 1: Fourier cosine series of $f(x) = \frac{\pi}{2}\sin(x)$ based on $[0,\pi]$

(ii) Since $\cos(n\pi) = (-1)^n$, we see that we can recover S_0 from $S(\frac{\pi}{2})$:

$$\frac{\pi}{2} = \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = S\left(\frac{\pi}{2}\right) = 1 - 2\sum_{n=1}^{\infty}\frac{\cos\left(2n\frac{\pi}{2}\right)}{4n^2-1} = 1 + 2S_0$$

$$\implies \qquad S_0 = \frac{\frac{\pi}{2}-1}{2} = \frac{\pi}{4} - \frac{1}{2}.$$

(iii) To prove uniform and absolute convergence we use the Weierstrass M-test. For every $x \in [-\pi, \pi]$ and $n \geq 1$, we can bound

$$\left| \frac{1}{4n^2 - 1} \cos(2nx) \right| \leq \frac{1}{3n^2} \leq \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the Weierstrass M-test we get uniform and absolute convergence of the series $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos(2nx)$ and therefore also for $S(x) = 1 - 2\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos(2nx)$.

Problem 3 The temperature of a rod with increasing thermal conductivity over time is modelled with the partial differential equation with boundary conditions

$$\left(\frac{1}{1+2t}\right) \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < \pi, \qquad t > 0, \tag{1}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi, t) = 0, \qquad t \geq 0. \tag{2}$$

a) Find all solutions of the form $u(x,t) = F(x)G(t)$ of (1) and (2).

b) Given the additional condition in $t = 1$

$$u(x,1) = 10\cos(x) + \sum_{n=2}^{\infty} \frac{\cos(nx)}{n^2} \qquad 0 \leq x \leq \pi, \tag{3}$$

find a solution $u(x,t)$ of (1), (2) and (3) for $t > 0$.

Solution.

a) 1) ODEs for F, G: Setting $u(x,t) = F(x)G(t)$, the PDE (1) is equivalent to

$$\frac{1}{1+2t} F(x)G'(t) - F''(x)G(t) = 0.$$

Assuming $F(x)G(t) \neq 0$ we can divide by $F(x)G(t)$ and get

$$\frac{1}{1+2t} \frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} = \text{const} = k,$$

since the left and right hand sides are functions of t and x respectively. Hence we get two ODEs

$$F''(x) = kF(x), \qquad G'(t) = (1+2t)kG(t).$$

From the boundary condition (2) we further get

$$F'(0)G(t) = u_x(0,t) = 0 = u_x(\pi, t) = F'(\pi)G(t)$$

so that by assuming $G \not\equiv 0$ we get

$$F'(0) = 0 = F'(\pi). \tag{4}$$

2) Solving for F: We solve for F depending on the different signs of k:

<u>$k = 0$</u>: The solution of $0 = F'' - kF = F''$ is $F(x) = Ax + B$. By (4) and $F'(x) = A$, we get $0 = F'(0) = F'(\pi) = A$, and hence $F(x) = B$.

<u>$k = c^2 > 0$</u>: The general solution of $0 = F'' - kF = F'' - c^2F$ is given by

$$F(x) = Ae^{cx} + Be^{-cx}.$$

By the boundary condition (4),

$$0 = F'(0) = c(A - B), \qquad 0 = F'(\pi) = c\left(Ae^c - Be^{-c}\right).$$

Then $A = B$ by he first equation ($c > 0$), and the seconds becomes $0 = cA\,(e^c - e^{-c})$. Since $e^c > 1, e^{-c} < 1$, $A = 0$ and hence $F \equiv 0$ and $u = FG = 0$.

<u>$k = -c^2 < 0$</u>: The general solution of $0 = F'' - kF = F'' + c^2F$ is given by

$$F(x) = A\cos(cx) + B\sin(cx).$$

Since $F'(x) = -Ac\sin(cx) + Bc\cos(cx)$, the boundary condition (4) yields

$$0 = F'(0) = Bc, \qquad 0 = F'(\pi) = -Ac\sin(c\pi) + Bc\cos(c\pi).$$

Since $c > 0$ the first equation yields $B = 0$, and then the second one gives $A = 0$ or $c = n \in \mathbb{Z}$. If $A = 0$, then $F \equiv 0$ and $u = FG = 0$. To have $u \neq 0$, we need $c = n$, $k = -n^2$ and then solution is $F(x) = A\cos(nx)$.

Conclusion: $F \neq 0$ only when $k = -n^2$ for $n \in \mathbb{Z}$ (including $n = 0$!), and then $F_n(x) = B_n \cos(nx)$ for any $B_n \in \mathbb{R}$ and $n \in \mathbb{N}$ or $n = 0$ ($F_{-n} = F_n$).

3) Solving for G when $F \neq 0$: I.e. when $k = -n^2$ for $n \in \mathbb{N}$ and $n = 0$. Either observe (and check!) that $G(t) = Ce^{k(t+t^2)}$ is a solution, or use the fact that the equation is separable (or an integrating factor)

$$\frac{\mathrm{d}G}{\mathrm{d}t} = (1+2t)kG(t) \quad \Longrightarrow \quad \int \frac{1}{G(t)}\,\mathrm{d}G = \int (1+2t)k\,\mathrm{d}t$$

$$\Longrightarrow \quad \ln(G(t)) = k(t+t^2) + c \quad \Longrightarrow \quad G(t) = Ce^{k(t+t^2)}.$$

Hence we find that $G_n(t) = C_n e^{-n^2(t+t^2)}$ for $C_n \in \mathbb{R}$ and $n \in \mathbb{N}$ and $n = 0$.

4) Conclusion: All solutions $u = F(x)G(t) \neq 0$ of (1) and (2) are given by

$$u_n(x,t) = F_n(x)G_n(t) = D_n \cos(nx) e^{-n^2(t+t^2)},$$

for $D_n \in \mathbb{R}$ and $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

b) Since (1) and (2) are linear homogeneous equations, we can use part a) and the superposition principle to conclude that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} D_n \cos(nx) e^{-n^2(t+t^2)},$$

is a solution of (1) and (2) (when it converges and is termwise differentiable).

By condition (3), we then have

$$\sum_{n=0}^{\infty} D_n \cos(nx) e^{-n^2(1+1^2)} = u(x,1) = 10\cos(x) + \sum_{n=2}^{\infty} \frac{\cos(nx)}{n^2}.$$

Comparing coefficients (which uniquely determine the series),

$$D_0 = 0, \qquad D_1 e^{-2} = 10, \qquad D_n e^{-2n^2} = \frac{1}{n^2} \quad \text{for } n > 1.$$

Hence the solution of (1) with boundary conditions (2) and (3) is given by

$$u(x,t) = 10e^{2-t-t^2}\cos(x) + \sum_{n=2}^{\infty} \frac{1}{n^2} e^{n^2(2-t-t^2)} \cos(nx).$$

Problem 4 Consider the functions

$$f(z) = iz - \frac{1}{z}, \qquad \text{and} \qquad g(z) = e^{-|z|^2}.$$

Show that f is analytic in $z \neq 0$, but g is nowhere analytic.

Solution. Let $z = x + iy$ and note that

$$f(z) = iz - \frac{1}{z} = iz - \frac{\overline{z}}{z\overline{z}} = i(x+iy) - \frac{x - iy}{x^2+y^2} = \underbrace{-y - \frac{x}{x^2+y^2}}_{=u(x,y)} + i\Big(\underbrace{x + \frac{y}{x^2+y^2}}_{=v(x,y)}\Big).$$

Now we will check the Cauchy-Riemann equations hold:

$$u_x(x,y) = -\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}, \qquad u_y(x,y) = -1 + \frac{2xy}{(x^2+y^2)^2},$$

$$v_y(x,y) = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}, \qquad v_x(x,y) = 1 - \frac{2xy}{(x^2+y^2)^2},$$

and hence $u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$ for all $z \neq 0$ (these terms are not defined at $z = 0$). Since all these partial derivatives of u, v are continuous, it follows that f is analytic for $z \neq 0$.

Simpler alternative: Compute the derivative f' from the definition:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{i(z+\Delta z) - \frac{1}{z+\Delta z} - \left(iz - \frac{1}{z}\right)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{i\Delta z - \frac{z-(z+\Delta z)}{z(z+\Delta z)}}{\Delta z} = \lim_{\Delta z \to 0} i + \frac{1}{z(z+\Delta z)} = i + \frac{1}{z^2}$$

for $z \neq 0$. Hence f' exists for all $z \neq 0$ and thus f is analytic in $\mathbb{C} \setminus \{0\}$.

Since $g(z) = e^{-|z|^2} = e^{-(x^2+y^2)} = u + iv$ for $v = 0$, and $e^{-(x^2+y^2)} > 0$,

$$\begin{aligned} u_x(x,y) &= -2xe^{-(x^2+y^2)} \neq 0 = v_y(x,y) \qquad &\text{for} \quad x \neq 0, \\ u_y(x,y) &= -2ye^{-(x^2+y^2)} \neq 0 = -v_x(x,y) \qquad &\text{for} \quad y \neq 0, \end{aligned}$$

Hence the Cauchy-Riemann equations only holds at $z = 0$, and not in any neighbourhood (or disk) in $\mathbb{C}$. Therefore f is nowhere analytic.

Problem 5 The Laurent series

$$\frac{e^{\frac{1}{z}}}{(z^2+4)(z-1)} = \sum_{n=0}^{\infty} a_n (z-1)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-1)^n}$$

converges in the point $z = 3$. For which of the points

$$z_1 = -2, \qquad z_2 = 1 + 2i, \qquad z_3 = \frac{1}{2} - \frac{1}{2}i$$

does it also converge? Justify your answer.

Solution. Let

$$f(z) = \frac{e^{\frac{1}{z}}}{(z^2+4)(z-1)} = \frac{e^{\frac{1}{z}}}{(z+2i)(z-2i)(z-1)}.$$

The numerator $e^{\frac{1}{z}}$ has only one singularity - an essential singularity at $z = 0$. Since the exponential function has no zeros, the zeros of the denominator are also singularities of f. Hence all singularities of f are given by $z_1^* = 0$, $z_2^* = 1$, $z_3^* = 2i$, and $z_4^* = -2i$.

The given Laurent series has centre $z_0 = 1$ (a singularity of f). By Laurent's theorem there exist unique convergent Laurent series centred at $z_0 = 1$ converging in the largest annulii where f is analytic. The inner and outer radii follow from computing the distance from centre to the singularities:

$$\begin{aligned} |z_0 - z_1| &= |1 - 0| = 1, \\ |z_0 - z_3| &= |1 - 2i| = \sqrt{1^2 + 2^2} = \sqrt{5}, \\ |z_0 - z_4| &= |1 + 2i| = \sqrt{5}. \end{aligned}$$

Hence we get three different Laurent series, each converging in one of the annuli

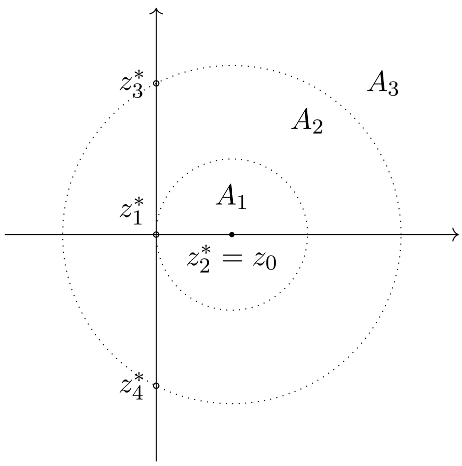
$$\begin{aligned} A_1 : &\quad 0 < |z - z_0| < 1, \\ A_2 : &\quad 1 < |z - z_0| < \sqrt{5}, \\ A_3 : &\quad \sqrt{5} < |z - z_0|. \end{aligned}$$

The series in this problem converges at $z = 3$, and

$$|z_0 - 3| = |1 - 3| = 2 \in \left(1, \sqrt{5}\right) \qquad \Longrightarrow \qquad 3 \in A_2$$

and the Laurent series converges in A_2. We then check convergence in z_1, z_2, z_3:

$$\begin{aligned} &|z_1 - z_0| = |-2 - 1| = 3 > \sqrt{5} && \Longrightarrow && z_1 \notin A_2, \text{ no convergence,} \\ &|z_2 - z_0| = |1 + 2i - 1| = 2 \in \left(1, \sqrt{5}\right) && \Longrightarrow && z_2 \in A_2, \text{ convergence in } z_2, \\ &|z_3 - z_0| = |\tfrac{1}{2} - \tfrac{1}{2}i - 1| = \sqrt{\tfrac{1}{2^2} + \tfrac{1}{2^2}} = \sqrt{\tfrac{1}{2}} < 1 && \Longrightarrow && z_3 \notin A_2, \text{ no convergence.} \end{aligned}$$



Problem 6

a) Let S_R be the half circle arc $z = Re^{i\theta}$ for $\theta \in [0, \pi]$. Show that

$$\int_{S_R} \frac{e^{iz}}{(z^2 + 4)^2} \, \mathrm{d}z \to 0 \quad \text{as} \quad R \to \infty.$$

b) Use the result from a) and residue calculation to calculate the value of the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + 4)^2} \, \mathrm{d}x.$$

Solution.

a) Let $f(z) = \frac{e^{iz}}{(z^2+4)^2}$. Then by the ML-inequality,

$$|I| \le \max_{z \in S_R} |f(z)| L,$$

where $L = \pi R$ is the length of S_R. If $z = x + iy \in S_R$, then $y \ge 0$, $|z| = R$,

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix-y}| = |e^{ix}|e^{-y} = e^{-y} \le 1 \qquad (y \ge 0),$$

and since $|z^2 + 4| \ge |z|^2 - 4$,

$$\frac{1}{|(z^2+4)^2|} = \frac{1}{|z^2+4|^2} \le \frac{1}{(|z|^2-4)^2} = \frac{1}{(R^2-4)^2} \qquad (|z| = R).$$

We conclude that $\max_{z \in S_R} |f(z)| \le \frac{1}{(R^2-4)^2}$ and

$$|I| \le \frac{\pi R}{(R^2-4)^2} \to 0 \qquad \text{as} \qquad R \to \infty.$$

b) The integral is equal to the principal part,

$$I = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)^2} \,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{(x^2+4)^2} \,\mathrm{d}x,$$

and then for any $R > 0$, taking $C_R = S_R \cup [-R, R]$ oriented counterclockwise,

$$\int_{-R}^{R} \frac{e^{ix}}{(x^2+4)^2} \,\mathrm{d}x = \oint_{C_R} \frac{e^{iz}}{(z^2+4)^2} \,\mathrm{d}z - \int_{S_R} \frac{e^{iz}}{(z^2+4)^2} \,\mathrm{d}z = I_1 - I_2.$$

We compute I_1 using the Residue theorem. Since e^{iz} is analytic and has no zeros, the poles of $f(z)$ are the zeros of the denominator

$$(z^2+4)^2 = ((z+2i)(z-2i))^2 = (z+2i)^2(z-2i)^2.$$

So $f(z)$ has order 2 poles at $z = \pm 2i$, and only $z = 2i$ is encircled by C_R for $R > 1$. Using the formula for residues of second order poles, we find that

$$\begin{aligned}\operatorname*{Res}_{z=2i} f(z) &= \lim_{z\to 2i} \frac{\mathrm{d}}{\mathrm{d}z}\left(\cancel{(z-2i)}^2 \frac{e^{iz}}{(z+2i)^2\cancel{(z-2i)}^2}\right) = \lim_{z\to 2i}\left(\frac{ie^{iz}}{(z+2i)^2} - 2\frac{e^{iz}}{(z+2i)^3}\right) \\ &= \frac{ie^{2i^2}}{(4i)^2} - 2\frac{e^{2i^2}}{(4i)^3} = \frac{e^{-2}}{-16}\left(i - 2\frac{1}{4i}\right) = -\frac{3i}{32e^2}.\end{aligned}$$

Let $R > 1$. Since C_R is a simple closed curve, and $f(z)$ is analytic on and inside C_R except at $z = 2i$, we can use the Residue theorem to conclude that

$$I_1 = 2\pi i \operatorname*{Res}_{z=2i} f(z) = \frac{3\pi}{16e^2}.$$

By part a) we have $\lim_{R\to\infty} I_2 = 0$, and can therefore conclude that

$$I = \lim_{R\to\infty} I_1 - \lim_{R\to\infty} I_2 = \frac{3\pi}{16e^2}.$$