# Fourier transform



General formulas:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega .$$

Now:

#1 a. $f_1(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & |t| > 1 \end{cases} \Rightarrow$

$$\hat{f}_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega t}\, dt = \sqrt{\frac{2}{\pi}}\, \frac{\sin \omega}{\omega}$$

b. $f_2(t) = \begin{cases} \sin t, & |t| \le 1 \\ 0, & |t| > 1 \end{cases} \Rightarrow$

$$\Rightarrow \hat{f}_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \frac{1}{2i}\left(e^{it} - e^{-it}\right) e^{-i\omega t}\, dt$$

$$= \frac{1}{2i}\left[\hat{f}_1(\omega - 1) - \hat{f}_1(\omega + 1)\right] =$$

$$= \frac{1}{i\sqrt{2\pi}} \left[\frac{\sin(\omega - 1)}{\omega - 1} - \frac{\sin(\omega + 1)}{\omega + 1}\right]$$

ⓒ $f_3(t) = \begin{cases} 1-|t| & -1 \le t \le 1 \\ 0, & |t|>1 \end{cases}$

$$\underbrace{\hat{f}_3(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwt}\,dt}_{\hat{f}_1(w)} - $$

$$- \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} (-t)\, e^{-iwt}\,dt - \frac{1}{\sqrt{2\pi}} \int_{0}^{1} t e^{-iwt}\,dt.$$

Each integral we evaluate separately :

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^{0} t e^{-iwt}\,dt = \frac{1}{\sqrt{2\pi}} \frac{1}{-iw} \int_{-1}^{0} t\, d e^{-iwt} =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-1}{iw} t e^{-iwt} \Big|_{-1}^{0} + \frac{1}{\sqrt{2\pi}} \frac{1}{iw} \int_{-1}^{0} e^{-iwt}\,dt =$$

$$= \frac{-1}{\sqrt{2\pi}} \frac{-1}{iw} (-1)\, e^{iw} - \frac{1}{\sqrt{2\pi}} \frac{1}{(iw)^2} e^{-iwt} \Big|_{-1}^{0} =$$

$$= \frac{-1}{\sqrt{2\pi}} \frac{1}{iw} e^{iw} - \frac{1}{\sqrt{2\pi}} \frac{1}{w^2} e^{iw} + \frac{1}{\sqrt{2\pi}} \frac{1}{w^2}$$

# Fourier transform 

$$-\frac{1}{\sqrt{2\pi}}\int_0^1 t\, e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}}\,\frac{1}{i\omega}\int_0^1 t\, d\, e^{-i\omega t} =$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{i\omega}\, e^{-i\omega} - \frac{1}{\sqrt{2\pi}}\,\frac{1}{i\omega}\int_0^1 e^{-i\omega t}\,dt =$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{i\omega}\, e^{-i\omega} + \frac{1}{\sqrt{2\pi}}\,\frac{1}{(i\omega)^2}\, e^{-i\omega t}\Big|_0^1 =$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{i\omega}\, e^{-i\omega} - \frac{1}{\sqrt{2\pi}}\,\frac{1}{\omega^2}\, e^{-i\omega} + \frac{1}{\sqrt{2\pi}}\,\frac{1}{\omega^2}$$

Finally

$$\hat{f}_3(w) = \cancel{\hat{f}_1(w) - \frac{2}{\sqrt{2\pi}}\,\frac{\sin w}{w}} -$$

$$-\frac{1}{\sqrt{2\pi}}\,\frac{1}{w^2}\left(e^{iw} + e^{-iw} - 2\right) =$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{1}{w^2}\left(\cos w - 1\right)$$

# Fourier transform

(d) $f_4(t) = e^{-2|t|}$

$$\hat{f}_4(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{2t - itw}\, dt +$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-2t - itw}\, dt =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2 - iw} e^{2t - itw} \Big|_{t=-\infty}^{0} +$$

$$+ \frac{1}{\sqrt{2\pi}} \frac{-1}{2 + iw} e^{-2t - itw} \Big|_{t=0}^{\infty} =$$

$$= + \frac{1}{\sqrt{2\pi}} \frac{1}{2 - iw} + \frac{1}{\sqrt{2\pi}} \frac{1}{2 + iw} =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{4}{4 + w^2}$$

#2 Inverse Fourier transform:

(a)
$$\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{\sin w}{w}\,e^{iwt}\,dw =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin w}{w}\,e^{iwt}\,dt = \begin{cases} 1, & |t|<1 \\ 0, & |t|>1 \end{cases}$$

(b)
$$\frac{1}{2i\pi}\int_{-\infty}^{\infty}\left[\frac{\sin(w-1)}{w-1} - \frac{\sin(w+1)}{w+1}\right]e^{+iwt}\,dw =$$

$$= \begin{cases} \sin t & |t|<1 \\ 0 & |t|>1 \end{cases}$$

(c)
$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos w}{w^2}\,e^{iwt}\,dt = \begin{cases} 1-|t|, & |t|<1 \\ 0, & |t|>1 \end{cases}$$

#3. Half-range expansions for

$$f(t) = e^{-t}, \quad t > 0.$$

1. Cos.-expansion:

Even prolongation: $f_e(t) = e^{-|t|}, \; -\infty < t < \infty$.

$$\hat{f}_e(w) = \frac{1}{\sqrt{2\pi}} \frac{1}{1+w^2}$$ (similarly to Problem #1, d).

$$f_e(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{1+w^2} e^{iwt} \, dw =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} \frac{1}{1+w^2} e^{iwt} \, dw + \frac{1}{2\pi} \int_{0}^{\infty} \frac{1}{1+w^2} e^{iwt} \, dw$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos wt}{1+w^2} \, dw$$

Odd prolongation:

$$f_o(t) = \begin{cases} e^{-t}, & t > 0 \\ -e^{t}, & t < 0 \end{cases}$$

# Fourier transform



$$\hat{f}_0(w) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{t} e^{-iwt} \, dt + $$

$$+ \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} e^{-iwt} \, dt$$

$$-\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{t-iwt} \, dt =$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{1}{1-iw} e^{t-iwt} \Big|_{t=-\infty}^{0} =$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{1}{1-iw} ;$$

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t-iwt} \, dt = \frac{1}{\sqrt{2\pi}} \frac{-1}{1+iw} e^{-t-iwt} \Big|_{t=0}^{\infty} =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{1+iw}$$

all together

$$\hat{f}_0(w) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{1+iw} - \frac{1}{1-iw} \right) =$$

$$= -i \sqrt{\frac{2}{\pi}} \frac{w}{1+w^2}$$

$$f_o(t) = \frac{-i}{\pi}\int_{-\infty}^{\infty} e^{i\omega t}\frac{\omega}{1+\omega^2}\,d\omega =$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\omega}{1+\omega^2}\sin\omega t\,d\omega$$

#4 Convolution and the Fourier transform of the functions

$$g(t) = u(t+1) - u(t-1), \quad h(t) = e^{-|t|}$$

We mention that $g(t)$ coincides with $f_1$ from Problem #1 and $h(t)$ coincides with $f_e$ from Problem #3.

Respectively

$$\hat{g}(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}$$

$$\hat{h}(\omega) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{1+\omega^2}$$

So

$$\mathcal{F}(g*h) = \sqrt{2\pi}\,\hat{g}\cdot\hat{h} = \sqrt{\frac{2}{\pi}}\,\frac{1}{1+\omega^2}\,\frac{\sin\omega}{\omega}$$

# Fourier transform



Evaluating the convolution

$$\int_{-\infty}^{\infty} g(\tau)\, h(t-\tau)\, d\tau = \int_{-1}^{1} e^{-|t-\tau|}\, d\tau$$

Various cases:

$\underline{\underline{t > 1}} \Rightarrow |t-\tau| = t-\tau$ for all $\tau$, $-1 < \tau < 1$

$$\Rightarrow \int_{-1}^{1} e^{-t+\tau}\, d\tau = e^{-t} \int_{-1}^{1} e^{\tau}\, d\tau = e^{-t}\left(e - \frac{1}{e}\right)$$

$\underline{\underline{t \leq -1}} \Rightarrow |t-\tau| = -t+\tau$

$$\Rightarrow \int_{-1}^{1} e^{t-\tau}\, d\tau = e^{t} \int_{-1}^{1} e^{-\tau}\, d\tau = e^{t}\left(e - \frac{1}{e}\right)$$

$\underline{\underline{-1 < t < 1}} \Rightarrow$

$$|t-\tau| = \begin{cases} t-\tau & \tau < t \\ \tau - t & \tau > t \end{cases}$$

$$\Rightarrow \int_{-1}^{1} e^{-|t-\tau|}\, d\tau = \int_{-1}^{t} e^{-t+\tau}\, d\tau + \int_{t}^{1} e^{t-\tau}\, d\tau =$$

$$= e^{-t}\left(e^{t} - e^{-1}\right) + e^{t}\left(e^{-t} - e^{-1}\right) =$$

$$= 1 - e^{-t-1} + 1 - e^{t-1} = 2 - e\left(e^{-t} + e^{t}\right)$$

Collecting all together.

$$g * h\,(t) = \begin{cases} 2e^{t}\sinh 1, & t < -1 \\ 2(1 - e\cosh t), & -1 < t < 1 \\ 2e^{-t}\sinh 1, & t > 1 \end{cases}$$